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ORIENTED OPEN-CLOSED STRING THEORY REVISITED

Barton Zwiebach*

Lyman Laboratory of Physics,
Harvard University
Cambridge, Massachusetts 02138, U.S.A.

ABSTRACT

String theory on D-brane backgrounds is open-closed string theory. Given the relevance of this fact, we give details and elaborate upon our earlier construction of oriented open-closed string field theory. In order to incorporate explicitly closed strings, the classical sector of this theory is open strings with a homotopy associative A_∞ algebraic structure. We build a suitable Batalin-Vilkovisky algebra on moduli spaces of bordered Riemann surfaces, the construction of which involves a few subtleties arising from the open string punctures and cyclicity conditions. All vertices coupling open and closed strings through disks are described explicitly. Subalgebras of the algebra of surfaces with boundaries are used to discuss symmetries of classical open string theory induced by the closed string sector, and to write classical open string field theory on general closed string backgrounds. We give a preliminary analysis of the ghost-dilaton theorem.

* E-mail address: zwiebach@irene.mit.edu
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1. Introduction and Summary

Recent developments indicate that the distinction between theories of open and closed strings and theories of closed strings is not fundamental. Theories that are formulated as pure closed string theories on simple backgrounds may show open string sectors on D-brane backgrounds (for a review, see [1]). In such backgrounds open-closed string field theory gives a complete description of the perturbative physics. It is therefore of interest to have a good understanding of open-closed string field theory. The elegant open string field theory of Witten [2] is formulated without including an explicit closed string sector. The price for this simplicity is lack of manifest factorization in the closed string channels [3,4]. It is now clear that it is generally useful to have an explicit closed string sector. A covariant open-closed string field theory achieving this was sketched in Ref.[5]. In this theory the Batalin Vilkovisky (BV) master equation is satisfied manifestly. One of the purposes of the present paper is to give the detailed construction promised in Ref.[5]. Much of that was actually written in 1991. In completing this paper now we have taken the oportunity to develop and explain some further aspects of open-closed theory. Since understanding the conceptual unity underlying closed and open-closed string theory is important, we will show that open-closed string theory and closed string theory are simply two different concrete realizations of the same basic geometrical and algebraic structures.

The structures include a set of spaces $\mathcal{P}$ where one can define an antibracket $\{\, ,\, \}$ and a delta operation Δ satisfying the axioms of the corresponding BV operators. From the spaces $\mathcal{P}$ one constructs a space $\mathcal{V}$ satisfying the condition $\partial\mathcal{V} + \frac{1}{2}\{\mathcal{V},\mathcal{V}\} + \Delta\mathcal{V} = 0$. Here ∂ is the boundary operator. In addition, one has a vector space $\mathcal{H}$, equipped with an antibracket and delta defined on $C(\mathcal{H})$ (functions on $\mathcal{H}$), and an odd quadratic function Q satisfying $\{Q, Q\} = 0$. Finally, there is a map f from $\mathcal{P}$ to $C(\mathcal{H})$ inducing a homomorphism between the respective BV structures, with the boundary operator ∂ realized by the hamiltonian Q. The action is $S = Q + f(\mathcal{V})$, and manifestly satisfies the BV master equation.

In the case of closed string field theory, where the above structures were recognized, $\mathcal{P}$ are moduli spaces of boundaryless Riemann surfaces having marked points (closed string punctures). The antibracket operation and delta are realized by twist-sewing [6,7], $\mathcal{H}$ is a suitably restricted state space of a conformal field theory, and the function Q is the BRST hamiltonian. The antibracket on $C(\mathcal{H})$ arises from a symplectic form in $\mathcal{H}$. The object $\mathcal{V}$ is defined as the formal sum of the string vertices $\mathcal{V}_{g,n}$ for all allowed values of g and n. Each string vertex $\mathcal{V}_{g,n}$ represents the region of the moduli space of Riemann surfaces of genus g and n punctures which cannot be obtained by sewing of lower dimension string vertices.

For open-closed string field theory the story, to be elaborated in this paper, goes as follows. The spaces $\mathcal{P}$ are moduli spaces of bordered Riemann surfaces having marked points in the interior (closed string punctures) and marked points on the boundaries (open string punctures). The antibracket and delta operation are realized by twist-sewing of closed string punctures and sewing of open string punctures. The vector space $\mathcal{H}$ is now of the form $\mathcal{H} = \mathcal{H}_o \oplus \mathcal{H}_c$, the direct sum of open and closed string sectors. The BRST operator on $\mathcal{H}$ is defined by the action of the open BRST operator Q_o on vectors

lying on $\mathcal{H}_o$ and by the action of the closed BRST operator Q_c on vectors lying on $\mathcal{H}_c$. The symplectic structure on $\mathcal{H}$ arises from separate symplectic structures on $\mathcal{H}_o$ and $\mathcal{H}_c$ without mixing of the sectors. The antibracket on $C(\mathcal{H})$ arises from the symplectic structure. The object $\mathcal{V}$ is defined as the formal sum of the string vertices $\mathcal{V}^{g,n}_{b,m}$ for all allowed values of g, n, b and m. Each string vertex $\mathcal{V}^{g,n}_{b,m}$ represents a region of the moduli space of Riemann surfaces of genus g, with n closed string punctures, b boundary components and a total of $m = m_1 + \cdots m_b$ open string punctures (m_i on the boundary component b_i). This vertex represents the region of the moduli space that cannot be obtained by sewing of lower dimensional vertices.

The above construction was not without some subtleties. Moduli spaces have to be Z_2 graded, and in closed string theory we used the dimension of the moduli space (mod 2) as a degree. This was consistent with the fact that the closed string antibracket, which involves twist sewing and thus adds one real dimension, is of degree one. For open strings, the antibracket involves just sewing, and does not increase dimensionality. We must therefore find a new definition of degree. It turns out that the proper definition is quite natural: the grade of a moduli space of surfaces $\mathcal{A}^{g,n}_{b,m}$ is the difference between its dimension and the canonical dimension of the moduli space $\mathcal{M}^{g,n}_{b,m}$ of surfaces of the same type. The antibracket can then be verified to be always of degree one. In order for the open string antibracket to have the correct exchange property the moduli spaces in $\mathcal{P}$ must satisfy additional conditions. Given a boundary component with m open string punctures, these punctures must be labelled cyclically, and the moduli space must go to itself, up to the sign factor $(-)^{m-1}$, under a cyclic permutation of the punctures. If there are several boundary components, these must also be labelled and moduli spaces must go into themselves up to nontrivial sign factors under the exchange of labels on the boundary components.

We thus see that open-closed string theory provides a new realization of the basic structures found in closed string theory. Open-closed string theory is therefore confirming the relevance of the structures we have learned about. Open-closed string theory is also suggesting the usefulness of bringing out explicitly each sector of the theory: it seems better to introduce a closed string sector rather than having it arise in a singular way. This may be an important lesson, especially on the light of recent developments that show the relevance of the soliton sectors in string theory. More sophisticated formulations of string theory may require some generalization of the above structures, as those discussed in Refs.[8,9] in the context of formulating string theory around non-conformal backgrounds. It is also possible that $\mathcal{P}$ spaces may satisfy the above axioms without being moduli spaces of surfaces. The possibility of defining $\mathcal{P}$ spaces from Grassmannians and using them to build string amplitudes has been studied in Ref.[10].

This paper begins (section 2) by discussing the moduli spaces of Riemann surfaces with boundaries and explaining the definition of local coordinates around open and closed punctures. We discuss the five possible sewing configurations and learn how to assign a Z degree to moduli spaces. We then turn to the definition of the antibracket and introduce the proper cyclic complex of surfaces. The discussion of the antibracket is facilitated by introducing a strictly associative multiplication of moduli spaces (section 2.5).

The string vertex $\mathcal{V}$ comprising the formal sum of all vertices of the open-closed string theory is introduced in section 3. We explain how the recursion relations take the form of a BV type master equation for $\mathcal{V}$. We give a derivation from first principles of the factors of $\hbar$ and coupling constant κ that accompany each moduli space $\mathcal{V}_{b,m}^{g,n}$ Taking three open string vertex to be at $\hbar^0$ and multiplied by a single power of κ, the $\hbar$ and κ dependence of all interactions is naturally fixed by the BV equation. The closed string kinetic term appears at the classical level and all other interactions at higher orders of $\hbar$.[⋆] At order $\hbar^{1/2}$, for example, we find the three-closed-string vertex and all the couplings of one closed string to $m \geq 0$ open strings through a disk. The string vertex corresponding to the moduli space of genus g surfaces with b boundary components, m open strings and n closed strings appears in order $\hbar^p$, where $p = 2g + b + \frac{1}{2}n - 1$. We also isolate useful subfamilies of vertices related by simple recursion relations. They are: (i) the set of all closed string vertices, (ii) the set of open string vertices on a disk, (iii) disks with open strings and one or zero closed strings, (iv) disks with open strings and two or less closed strings, (v) disks with all numbers of open and closed strings.

In section 4 we review the minimal area problem that defines the string diagrams of open-closed string theory and extract all string vertices corresponding to moduli spaces up to real dimension two. Here we discuss the classical open string sector of the open-closed theory. We explain why the strictly associative version of classical open string theory cannot incorporate an explicit closed string sector. The algebraic structure of the classical open string sector is that of an A_∞ algebra [14] equipped with additional structure, as explained in [15]. In section 5 we discuss the state spaces of open-closed conformal theories and the symplectic structures. We give the map from moduli spaces to string functionals, write the master action and verify that it satisfies the master equation.

The explicit description of all vertices coupling open and closed strings through a disk is the subject of section 6. We give some comments of the ways of describe boundary states $\langle B|$ both as states in the dual state space $\mathcal{H}_c^*$ or as states in $\mathcal{H}_c$. The open-closed vertex is discussed, noting in particular that it induces maps between the BRST cohomologies of the closed and open sectors. These maps take open string cohomology at ghost number G to closed string cohomology at $G+2$, and closed string cohomology at ghost number G to open string cohomology at ghost number G. We give the string diagrams for the open-open-closed vertex and show how it is generalized to the case of M open strings coupling to a single closed string. Then we give the string diagram for the open-closed-closed vertex and use it to generalize to the case of M open strings coupling to two closed strings. Finally, we extend this to the case of M open strings and N closed strings coupling through a disk.

⋆ The same seems to hold in light-cone open-closed string field theory. The Lorentz invariance of the classical open string theory and the classical closed string theory was proven in refs.[11]. The open-closed theory was studied in [12] and shown *not* to be invariant under the expected Lorentz transformation. Kikkawa and Sawada [13] have shown that the non-invariance of the action is actually cancelled by the non-invariance of the measure. Introducing the relevant factors of $\hbar$ in [13] the orders of $\hbar$ of all interactions are fixed once the open string theory is considered classical. The closed string kinetic term must be classical, and the open-closed and open-open-closed interactions appear at order $\hbar^{1/2}$ as they did in the present work.

Following the early work of Hata and Nojiri [16], we discuss in section 7 classical open string symmetries that arise from the closed string sector. In this symmetry transformation, the inhomogeneous part of the open string field shift is along the image, under the cohomology map mentioned above, of the ghost number one closed string cohomology. We elucidate completely the algebra of such transformations. All these facts follows easily from analysis of the subalgebras of surfaces involving one or two closed strings coupling to any number of open strings. We then use the subalgebra of all couplings of open and closed strings through disks to construct in section 8 a gauge invariant classical open string theory describing propagation on a nontrivial closed string background. By the nature of the subalgebras of surfaces, doing the opposite, *i.e.* closed string propagation on open string backgrounds, seems difficult to achieve.

This paper concludes in section 9 with a brief discussion of the issues that arise in establishing a ghost dilaton theorem in the open-closed system, and related issues in background independence. For the ghost dilaton theorem a few preliminary observations are the following. While the main effect of changing the string coupling is due to the shift of the closed string field along the ghost-dilaton, an inhomogeneous shift of the open string field along an unphysical direction appears to be necessary. This relies on the fact that the image of the ghost-dilaton under the open-closed cohomology map is a trivial state. Moreover, a non-vanishing coupling to a boundary of the ghost-dilaton indicates the presence of a tree level cosmological term associated to the open-string partition function on the disk.

2. Moduli spaces and their BV algebra

In the present section we will begin by discussing a few of the basic properties of the moduli spaces of Riemann surfaces with boundaries and punctures. The punctures can be open string punctures, if they are located at boundaries, and closed string punctures, if they are located in the interior of the surface. We then define the sewing of surfaces, and briefly discuss the five inequivalent sewing operations that can be performed on surfaces with open and closed string punctures.

In order to define an antibracket and a delta operator (of BV type) acting on moduli spaces we introduce a Z degree on moduli spaces. In addition we introduce a strictly associative multiplication of moduli spaces. We show that the moduli spaces must satisfy cyclicity conditions: up to well defined sign factors, the moduli spaces are invariant under the operation of cyclic permutation of the labels of the punctures lying on any boundary component. We define the antibracket and verify it satisfies all the desired properties. This is an extension of the definition of a BV algebra for the moduli spaces of surfaces without boundaries [6,7]. Some nodding familiarity with the discussion of Ref.[6] will be assumed.

2.1. Moduli spaces for oriented open-closed string theory

The general moduli space is that of (oriented) Riemann surfaces of genus $g \geq 0$, with $n \geq 0$ labelled interior punctures, representing closed string insertions and $b \geq 0$ boundary components. At the boundary components there are labelled punctures representing open string insertions. Let $m_i \geq 0$ denote the number of punctures at the i-th boundary component. The total number of open string punctures is therefore $m = \sum_{i=1}^{b} m_i$. Except for some low dimensional cases the (real) dimensionality of the moduli space $\mathcal{M}_{b,m}^{g,n}$ is given by

$$\mathrm{dim}\mathcal{M}_{b,m}^{g,n} = 6g - 6 + 2n + 3b + m\,. \tag{2.1}$$

All punctures must be equipped with analytic local coordinates (see Fig. 1) As usual the closed string punctures are equipped with local coordinates defined only up to phases. These are simply defined via an analytic map of a unit disk into the surface, with the origin of the disk going to the closed string puncture. There is no natural way to fix the phase of the local coordinate at an interior puncture.

The local coordinates for the open string punctures are defined as follows (Fig. 1). An open string coordinate is an analytic map from the upper half-disk $\{|w| \leq 1, \mathrm{Im}(w) \geq 0\}$ into a neighborhood of the puncture, with the origin going to the puncture and the boundary $\{\mathrm{Im}(w) = 0\}$ of the half-disk going into the boundary of the Riemann surface. There is no phase ambiguity in this definition. The real axis of the local coordinate coincides with the boundary, and positive imaginary values are inside the surface. Note that the orientation of the Riemann surface, defined by the usual orientation on every complex chart, induces an orientation on the boundary components, an orientation that can be pictured as an arrow. As we travel along the oriented boundary we always move along increasing real values for the local coordinates.

It will be useful to have an ordered list of the lowest dimensional moduli spaces of punctured surfaces with boundary components. We do not assign a negative contribution to the dimensionality arising from conformal Killing vectors. Our list begins with dimension zero moduli spaces.

<u>Dimension zero</u>

–the sphere with zero$^{*\,6}$, one$^{*\,4}$, two$^{*\,2}$, or three closed string punctures;

–the disk with zero$^{*\,3}$, one$^{*\,2}$, two$^{*\,1}$, or three open string punctures;

–the disk with one$^{*\,1}$ closed string puncture, and,

–the disk with one open and one closed string puncture.

The surfaces followed by an asterisk as in $\{\ \}^{*\,n}$ have n real conformal Killing vectors.

<u>Dimension one</u>

–the disk with four open punctures,

–the disk with one closed and two open punctures,

–the disk with two closed punctures, and,

–the annulus with zero$^{*\,1}$ or one open string puncture.

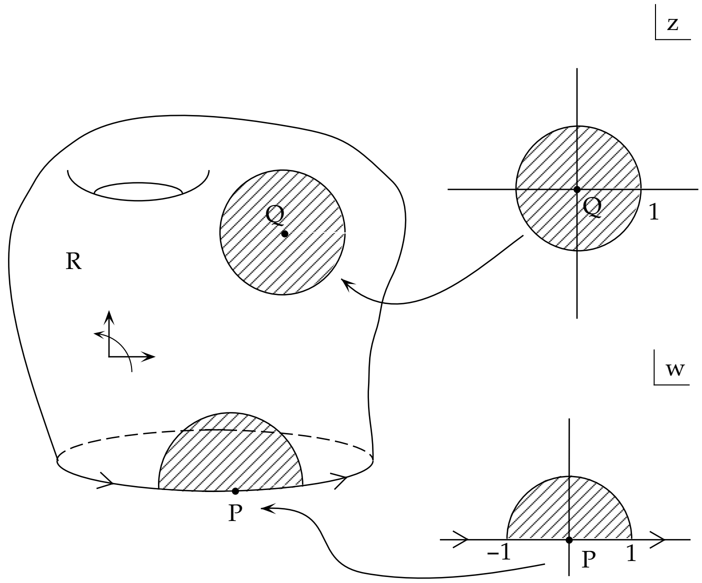


Figure 1. A Riemann surface R with a closed string puncture at Q and an open string puncture at P. The local coordinate z at Q is defined by an analytic map from the unit disk to the surface. The local coordinate w at P is defined by an analytic map of the upper-half-disk into the surface. The real line in the upper-half-disk maps into the boundary of the surface R.

Dimension two

–the torus with zero^{*2} or one puncture,

–the four punctured sphere,

–the disk with five open punctures,

–the disk with one closed and three open punctures,

–the disk with two closed and one open puncture,

–the annulus with two open punctures (two cases), and,

–the annulus with one closed puncture.

For the case of the annulus with two open punctures, the two punctures may lie in the same boundary component, one puncture may lie in each boundary component.

2.2. Sewing of surfaces and moduli spaces

An interaction vertex is represented by a blob (Fig. 2) and will be accompanied by the data (g, n, b, m). Only if needed explicitly we will give the number of open strings at each boundary component. Wavy lines emerging from the blob represent closed strings, each heavy dot represents a boundary component, and the straight lines emerging from them are open strings .

The canonical sewing operation for open strings is described by an identification of the type $zw = t$, where z and w are two local coordinates defined for open string punctures and t is a constant. Note that this constant should be real so that the boundaries of the

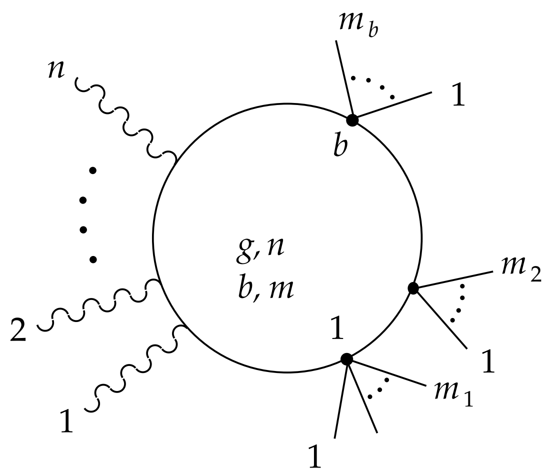


Figure 2. The representation of a string vertex arising from the moduli space of surfaces of genus g, with n closed string punctures, b boundary components, and a total of m open string punctures. The heavy dots represent boundary components. Straight lines are open string punctures and wavy lines represent closed string punctures. The i-th boundary component has m_i punctures, and $m = \sum_{i=1}^{b} m_i$.

half-disks defined by the local coordinates are glued completely into each other. Moreover, given our definition of the half disks as corresponding to the region of the disk with positive imaginary part, the point $z = i$ could only be glued with the point $w = i$, and this requires that the constant be equal to minus one. Therefore, the canonical sewing operation for open strings is of the form

$$zw = -1\,.$$

For closed strings, one usually takes the canonical sewing operation to be defined as $zw = 1$ where z and w are local coordinates around closed string punctures.

Consider a single sewing operation. It may involve two surfaces or it may involve a single surface. Moreover, the sewing operation may be the sewing of two open string punctures or the sewing of two closed string punctures. We want to verify that whenever the sewing is of open string punctures the resulting surface belongs to a moduli space whose real dimensionality is greater by one unit than that of the moduli space (s) of the original surface (s). For closed string sewing, the resulting surface belongs to a moduli space whose real dimensionality is greater by two units than that of the moduli space (s) of the original surface (s). An obvious consequence of the above statements is very familiar. Sewing of open string punctures with one real variable sewing parameter, and sewing of closed string punctures with two real sewing parameters, are both operations that starting with moduli spaces of proper dimensionality give moduli spaces of proper dimensionality.

Let us first consider sewing of open string punctures. Here there are three possibilities:

(i) The open string sewing joins two surfaces. In this case the genus and the number of closed string punctures simply add. The total number of boundaries is decreased by one, and the total number of open string punctures is decreased by two. One readily verifies

using (2.1) that

$$\mathrm{dim}\mathcal{M}^{g_1+g_2\,,\,n_1+n_2}_{b_1+b_2-1,m_1+m_2-2} = \mathrm{dim}\mathcal{M}^{g_1,n_1}_{b_1,m_1} + \mathrm{dim}\mathcal{M}^{g_2,n_2}_{b_2,m_2} + 1\,. \tag{2.2}$$

(ii) The open string sewing joins two open string punctures lying on the same boundary component of a single surface. In this case the genus and the number of closed string punctures do not change. The total number of boundaries is increased by one, and the total number of open string punctures is decreased by two. Again, one verifies that

$$\mathrm{dim}\mathcal{M}^{g\,,\,n}_{b+1,m-2} = \mathrm{dim}\mathcal{M}^{g,n}_{b,m} + 1\,. \tag{2.3}$$

(iii) The open string sewing joins two open string punctures lying on different boundary components of a single surface. This operation acturally increases the genus by one unit and decreases the number of boundaries by one unit. One can understand this as follows. Gluing completely two boundaries adds handle; partial gluing of two boundaries is the same as adding a handle with a hole. Gluing of open string punctures is indeed partial gluing of boundaries, thus explaining why the genus increases by one, while the number of boundaries decreases by one. A short calculation confirms that

$$\mathrm{dim}\mathcal{M}^{g+1\,,\,n}_{b-1,m-2} = \mathrm{dim}\mathcal{M}^{g,n}_{b,m} + 1\,. \tag{2.4}$$

For the case of sewing of closed string punctures there are only two configurations:

(iv) Closed string sewing of two surfaces. Here the number of boundaries and the number of open string punctures simply add. The genus adds, and the total number of closed string punctures is reduced by two. One readily finds that

$$\mathrm{dim}\mathcal{M}^{g_1+g_2\,,\,n_1+n_2-2}_{b_1+b_2\,,\,m_1+m_2} = \mathrm{dim}\mathcal{M}^{g_1,n_1}_{b_1,m_1} + \mathrm{dim}\mathcal{M}^{g_2,n_2}_{b_2,m_2} + 2\,. \tag{2.5}$$

(v) Closed string sewing involving a single surface. Here the number of boundaries and the number of open string punctures remain the same. The genus is increased by one unit, and the total number of closed string punctures is reduced by two units. One readily confirms that

$$\mathrm{dim}\mathcal{M}^{g+1,n-2}_{b\,,\,m} = \mathrm{dim}\mathcal{M}^{g,n}_{b,m} + 2\,. \tag{2.6}$$

2.3. Assigning a Z degree to moduli spaces

In order to build a Batalin-Vilkovisky algebra whose elements are moduli spaces of surfaces we must assign a Z_2 grading to the moduli spaces. In the closed string case this Z_2 grading was given by the dimensionality of the moduli space in question. The antibracket operation of two surfaces amounted to twist sewing (sewing with $zw = \exp(i\theta), 0 < \theta < 2\pi$) of two closed string punctures one in each surface. The antibracket of two moduli spaces is the set of surfaces obtained by taking the antibracket of every surface in the first moduli space with every surface in the second moduli space. Since twist sewing has one real parameter, it gives a moduli space whose dimension is one unit bigger than the sum of the dimensions of the moduli spaces that are to be sewn. This is exactly what one wishes, since, by definitiion, the antibracket is an operation with odd degree.

This brings in a first puzzle. It is quite clear from experience with open string theory that the antibracket on the open string sector can only amount to the sewing of two open string punctures lying on two different surfaces. In this case the antibracket would not change dimensions and would seem to correspond to an operation of even degree. It is clear that we need a new definition of the degree associated to a moduli space of surfaces.

Let us first notice that in the closed string case there was one important fact: all moduli spaces of proper dimensionality had even degree. This was easily achieved since the dimension of $\mathcal{M}^{g,n}$ is always even. We wish to have the same property for open-closed moduli spaces of proper dimensionality. This cannot be achieved by setting the degree equal to the dimension, since the dimension of $\mathcal{M}^{g,n}_{b.m}$ can be odd. The way out is clear, for any given moduli space $\mathcal{A}^{g,n}_{b,m}$ we define the Z degree ϵ to be given by

$$\epsilon(\mathcal{A}^{g,n}_{b,m}) = \text{dim}\mathcal{M}^{g,n}_{b.m} - \text{dim}\mathcal{A}^{g,n}_{b.m}\,. \tag{2.7}$$

Note that for moduli spaces of closed surfaces this degree induces exactly the same Z_2 degree we had before. Moreover, by definition, proper dimensionality moduli spaces now have degree zero. If we only need a Z_2 degree, many of the terms in (2.1) are irrelevant and we find

$$\epsilon(\mathcal{A}) = b + m + \text{dim}\mathcal{A}\,. \tag{2.8}$$

More important, we can now verify easily that the antibracket will have degree equal to $+1$ both in the open and in the closed string sector. Consider two moduli spaces $\mathcal{A}_1$ and $\mathcal{A}_2$ whose antibracket we are computing in the open string sector. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ denote the canonical moduli spaces associated to $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively. Moreover, let $\mathcal{M}_{12}$ denote the canonical moduli space associated to the surfaces that appear in the open string antibracket $\{\mathcal{A}_1, \mathcal{A}_2\}_o$. Since open string sewing adds no dimension, the dimension of $\{\mathcal{A}_1, \mathcal{A}_2\}_o$ is just the sum of the dimensions of $\mathcal{A}_1$ and $\mathcal{A}_2$. It then follows that

$$\begin{aligned}\epsilon(\{\mathcal{A}_1, \mathcal{A}_2\}_o) &= \text{dim}\mathcal{M}_{12} - (\text{dim}\mathcal{A}_1 + \text{dim}\mathcal{A}_2),\\ &= \text{dim}\mathcal{M}_1 + \text{dim}\mathcal{M}_2 + 1 - \text{dim}\mathcal{A}_1 - \text{dim}\mathcal{A}_2,\\ &= (\text{dim}\mathcal{M}_1 - \text{dim}\mathcal{A}_1) + (\text{dim}\mathcal{M}_2 - \text{dim}\mathcal{A}_2) + 1,\\ &= \epsilon(\mathcal{A}_1) + \epsilon(\mathcal{A}_2) + 1,\end{aligned} \tag{2.9}$$

where use was made of (2.2). A rather analogous computation gives the same exact result for closed string antibracket $\{\mathcal{A}_1, \mathcal{A}_2\}_c$. This time, twist sewing implies that the dimension of $\{\mathcal{A}_1, \mathcal{A}_2\}_c$ is equal to (dim $\mathcal{A}_1$+ dim$\mathcal{A}_2 + 1$), while $\text{dim}\mathcal{M}_{12} = \text{dim}\mathcal{M}_1 + \text{dim}\mathcal{M}_2 + 2$, by virtue of (2.5).

We can also verify that the delta operation Δ is also an operation of degree equal to $+1$. Recall that in the closed string sector Δ twist sews two closed string punctures lying on the same closed surface. In the open string sector the delta operator becomes an operator Δ_o that will simply sew two open string punctures lying on the same surface.*

$\star$ The BV algebra of surfaces can be described taking Δ to be the fundamental operation, as done for closed Riemann surfaces in [7]. In here we will describe explicitly the antibracket, and only describe Δ qualitatively.

Denoting by $\mathcal{M}$ the canonical moduli space associated to $\mathcal{A}$ and by $\Delta_o\mathcal{M}$ the canonical moduli space associated to $\Delta_o\mathcal{A}$ we have

$$\begin{aligned}\epsilon(\Delta_o\mathcal{A}) &= \mathrm{dim}\Delta_o\mathcal{M} - \mathrm{dim}\Delta_o\mathcal{A},\\ &= \mathrm{dim}\mathcal{M} + 1 - \mathrm{dim}\mathcal{A},\\ &= (\mathrm{dim}\mathcal{M} - \mathrm{dim}\mathcal{A}) + 1,\\ &= \epsilon(\mathcal{A}_1) + 1,\end{aligned} \tag{2.10}$$

where use was made of either (2.3) or (2.4). An identical equation holds for the closed string sector of the delta operation. Finally, the boundary operator, which decreases the dimensionality of a moduli space by one unit, also has degree +1

$$\epsilon(\partial\mathcal{A}) = \epsilon(\mathcal{A}) + 1\,, \tag{2.11}$$

as is clear from the definition of degree. All in all we have found a definition of the Z degree of moduli spaces such that the operations $\{\cdot,\cdot\}$, Δ and ∂ have all degree equal to plus one. Moreover, moduli spaces of proper dimensionality are all of degree zero, and thus correspond to even elements of the algebra.

2.4. The cyclic complex $\mathcal{P}$ of moduli spaces

While we know from the previous subsection what is roughly the definition of the antibracket, we must define carefully its action in order to verify that it satisfies the correct exchange property

$$\{\mathcal{A}_1, \mathcal{A}_2\} = -(-)^{(\mathcal{A}_1+1)(\mathcal{A}_2+1)}\,\{\mathcal{A}_2, \mathcal{A}_1\}\,, \tag{2.12}$$

as well as the Jacobi identity

$$(-)^{(\mathcal{A}_1+1)(\mathcal{A}_3+1)}\,\{\mathcal{A}_1, \{\mathcal{A}_2, \mathcal{A}_3\}\,\} + \text{cyclic} = 0\,. \tag{2.13}$$

Our strategy will be to consider first the case when the moduli spaces contain surfaces having a single boundary component and we will look at the antibracket in the open string sector. Both $\mathcal{A}_1$ and $\mathcal{A}_2$ are oriented, with orientations defined by a set of basis vectors $[\mathcal{A}_1]$ containing dim $(\mathcal{A}_1)$ vectors, and a set of basis vectors $[\mathcal{A}_2]$ containing dim $(\mathcal{A}_2)$ vectors, respectively. We will define the orientation of the antibracket $\{\mathcal{A}_1, \mathcal{A}_2\}_o$ by the ordered set of vectors $[[\mathcal{A}_1], [\mathcal{A}_2]]$. We face a puzzle in trying to reproduce (2.12). It seems faily clear that the antibracket $\{\mathcal{A}_1, \mathcal{A}_2\}_o$ and the antibracket $\{\mathcal{A}_2, \mathcal{A}_1\}_o$ could only be related by a sign factor involving the product of the dimensions of the respective moduli spaces. If this is the only sign factor, one cannot reproduce the sign factor involving the product of the degrees, since the degrees involve the number of open string punctures, as shown in (2.8).

The solution to this complication is interesting. We have to work in a complex where the moduli spaces have suitable properties under a change of labels associated to the the open string punctures. Indeed, in the closed string sector, moduli spaces are assumed to be invariant under any permutation of labels associated to closed string punctures. For

moduli spaces of bordered surfaces we introduce a *cyclic complex* $\mathcal{P}$, where the word cyclic refers to a property under the cyclic permitation of open string punctures on a boundary component. Consider a surface having a single boundary component and m open string punctures located at this boundary. The punctures will be labelled in cyclic order, namely, as we go around the oriented boundary the labels of the punctures we encounter always increase by one unit (mod m). Let $\mathcal{C}$ denote the operator that acting on a surface Σ gives the surface $C\Sigma$ which differs from Σ by a cyclic permutation of the labels of the punctures. After the action of C the puncture that used to have the label 1 has the label 2, the puncture that had label 2 now is labeled 3 and so on. If we denote by $l_P(\Sigma)$ the label of the point P in Σ, this means that

$$l_P(C\Sigma) = l_P(\Sigma) + 1 \quad [\mathrm{mod}\, m]\,. \tag{2.14}$$

Acting on a moduli space of surfaces the operator C will do a cyclic permutation of the punctures in every surface of the moduli space. It is clear that by definition acting on a space with m punctures $C^m = 1$. A moduli space $\mathcal{A}$ of surfaces having a single boundary component and m punctures is said to belong to the cyclic complex $\mathcal{P}$ if

$$C\mathcal{A} = (-)^{m-1}\mathcal{A}\,, \quad \forall \mathcal{A} \in \mathcal{P}\,. \tag{2.15}$$

Namely, under a cyclic permutation of the labels, the moduli space goes to itself except for the sign factor $(-)^{m-1}$. This nontrivial sign factor will play a role in giving the correct exchange property for the antibracket. Note that (2.15) is consistent with $C^m\mathcal{A} = \mathcal{A}$. The sign factor for a cyclic step is the sign factor that would arise if we do the cyclic step by successive exchanges of pairs of labels, and we assume that each exchange carries a factor of minus one. This is in accord with the idea that the open string field is naturally odd in some formulations of open string theory. In the present formulation, where the open string field is even, the odd property is carried by the complex of moduli space. Indeed, the symplectic form, associated to a two-punctured disk, is naturally odd in the present formulation. Given a moduli space $\mathcal{A}$ having no particular cyclicity property we will denote by

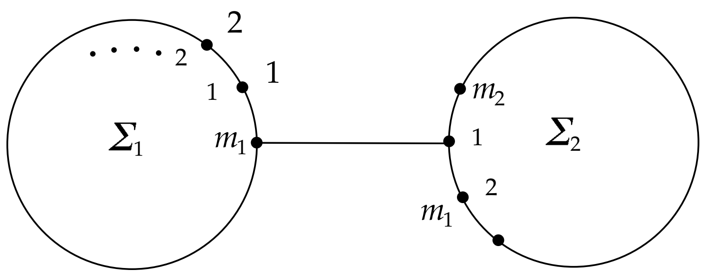
$$(\mathcal{A})_{cyc} \equiv \sum_{i=1}^{m-1} (-)^{i(m-1)}\, C^i \mathcal{A} \tag{2.16}$$

the cyclic moduli space that arises by explicit addition of $m-1$ cyclic copies of $\mathcal{A}$ weighted with the appropriate sign factors. It is simple to verify that $(\mathcal{A})_{cyc}$ satisfies (2.15).

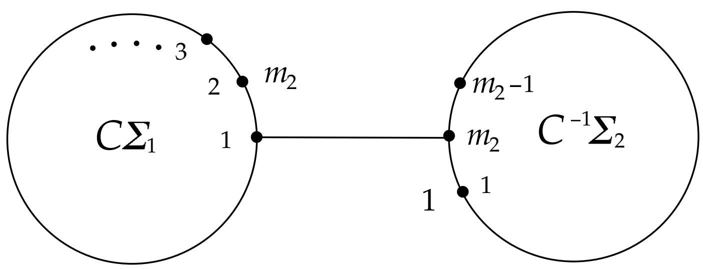
2.5. An associative multiplication

It will be helpful to introduce a multiplication of surfaces (each having a single boundary component, for simplicity). Given a surface Σ_1 and a surface Σ_2 we define the product $\Sigma_1 \circ \Sigma_2$ as the surface obtained by gluing the last puncture of Σ_1 to the first puncture of Σ_2. The remaining punctures are relabeled cyclically by keeping the original labels in the remaining punctures in Σ_1 and extending this to the glued surface. It is important to notice that the first puncture of $\Sigma_1 \circ \Sigma_2$ is the first puncture of Σ_1, and the last puncture of $\Sigma_1 \circ \Sigma_2$ is the last puncture of Σ_2. This multiplication is not commutative, but it is

clearly strictly associative. This multiplication is extended in the obvious way to a multiplication of moduli spaces, preserving associativity. The orientation of the moduli space $\mathcal{A} \circ \mathcal{B}$ is defined by the set of vectors $[[\mathcal{A}][\mathcal{B}]]$ where $[\mathcal{A}]$ and $[\mathcal{B}]$ are the vectors defining the orientations of $\mathcal{A}$ and $\mathcal{B}$ respectively. Note, however, that if two moduli spaces are cyclic, the product will not be. This is clear, because in all of the resulting surfaces the last puncture always lies on the part of the boundary component that originated from surfaces in the second moduli space, and it is always one step away from the place where the sewing operation takes place. This is clearly not a cyclic moduli space. I have not found a way to define an associative multiplication in the cyclic complex.



(a)



(b)

Figure 3. (a) The product $\Sigma_1 \circ \Sigma_2$ joins the last puncture of Σ_1 (label m_1) to the first puncture of Σ_2. The labels inside the blobs are the original labels of the punctures, the labels outside the blobs are the labels after gluing. (b) The product $C^{-1}\Sigma_2 \circ C\Sigma_1$. Note that the final labels of the punctures in cases (a) and (b) can be made to agree by doing $m_2 - 1$ cyclic permutations of the punctures in (b).

While the multiplication is not commutative, by suitable cyclic permutations of the punctures we can obtain an equality relating the two different ways of multiplying two surfaces. We claim that

$$\Sigma_1 \circ \Sigma_2 = (C^{-1})^{m_2-1}\big(C^{-1}\Sigma_2 \circ C\Sigma_1\big) \tag{2.17}$$

This property is best explained making use of the Fig.3. Note that in the left hand side we use the last puncture of Σ_1 while the operation on the right hand side would use the first puncture of $C\Sigma_1$. This is as it should be since these punctures are really the same concrete"physical" puncture (see (2.14)). Similarly, the first puncture of Σ_2 is the same puncture as the last puncture of $C^{-1}\Sigma_2$. The overall cyclic factor $(C^{-1})^{m_2-1}$ in the right

hand side is necessary because the labels of the resulting surfaces should also agree. In the left hand side the first puncture of the glued surface is the first puncture of Σ_1, while in the right hand side the same puncture carries the label m_2. It follows that $m_2 - 1$ anti-cyclic permutations of the right hand side are necessary for these labels to coincide. This concludes the verification of equation (2.17). When we deal with moduli spaces, there is an extra sign factor in the above relation. We have that

$$\mathcal{A}_1 \circ \mathcal{A}_2 = (-)^{d_1 d_2} (C^{-1})^{m_2 - 1} \big(C^{-1} \mathcal{A}_2 \circ C \mathcal{A}_1 \big) \,, \tag{2.18}$$

where d_1 and d_2 denote the dimensions of the moduli spaces $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively. This sign factor arises because of the way we defined the orientation for the product of two moduli spaces. If the moduli spaces involved are cyclic, that is, $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{P}$, the above exchange property simplifies considerably

$$\mathcal{A}_1 \circ \mathcal{A}_2 = (-)^{d_1 d_2 + m_1 + m_2} (C^{-1})^{m_2 - 1} \big(\mathcal{A}_2 \circ \mathcal{A}_1 \big) \tag{2.19}$$

where the extra factors arise from the action of C and C^{-1} on the cyclic moduli spaces. In order to find an equation that does not involve explicitly the operator C we can now form the cyclic moduli spaces as in equation (2.16) to find

$$(\mathcal{A}_1 \circ \mathcal{A}_2)_{cyc} = (-)^{d_1 d_2 + m_1 + m_2 + (m_2 - 1)(m_1 + m_2 - 3)} \big(\mathcal{A}_2 \circ \mathcal{A}_1 \big)_{cyc} \,, \tag{2.20}$$

where the new sign factor arises because the moduli space $(\mathcal{A}_2 \circ \mathcal{A}_1)_{cyc}$ is cyclic and its surfaces have $(m_1 + m_2 - 2)$ punctures. The sign factor can be simplified to read

$$(\mathcal{A}_1 \circ \mathcal{A}_2)_{cyc} = -(-)^{d_1 d_2 + m_1 m_2} \big(\mathcal{A}_2 \circ \mathcal{A}_1 \big)_{cyc} \,. \tag{2.21}$$

Note that this exchange property involves not only the dimensionalities of the moduli spaces but also the numbers of punctures. This is the kind of result that we needed for the antibracket, since the degree of a moduli space involves both the dimension of the space and the number of open string punctures.

2.6. Defining the antibracket

The exchange property of the antibracket, given in (2.12) requires a sign factor that differs slightly from that in (2.21). The desired sign factor arises if we define the antibracket (in the open string sector) as

$$\{\mathcal{A}_1, \mathcal{A}_2\}_o \equiv (-)^{m_1 d_2} \, (\mathcal{A}_1 \circ \mathcal{A}_2)_{cyc} \,. \tag{2.22}$$

It is now a simple computation using (2.21) to verify that

$$\{\mathcal{A}_1, \mathcal{A}_2\}_o = -(-)^{(d_1 + m_1)(d_2 + m_2)} \{\mathcal{A}_2, \mathcal{A}_1\}_o \,. \tag{2.23}$$

Using (2.7) and (2.1) we see that the Z_2 degree of a moduli space with one boundary component is $\epsilon(\mathcal{A}) = m + d_{\mathcal{A}} + 1$. It then follows that (2.23) is the correct exchange property for the antibracket.

The antibracket must satisfy a Jacobi identity of the form indicated in (2.13). Consider the first term in this identity (always in the open string sector of the antibracket). Using (2.22) we find that

$$(-)^{(\mathcal{A}_1+1)(\mathcal{A}_3+1)}\,\{\mathcal{A}_1,\{\mathcal{A}_2,\mathcal{A}_3\}\,\} = (-)^{s_{13}}\big(\mathcal{A}_1\circ(\mathcal{A}_2\circ\mathcal{A}_3)_{cyc}\big)_{cyc}\,, \tag{2.24}$$

where the sign factor s_{13} is given by

$$s_{13} = d_1d_3 + m_1m_3 + (d_1m_3 + d_2m_1 + d_3m_2)\,.$$

On the other hand, the last term of the Jacobi identity, for example, would read

$$(-)^{(\mathcal{A}_3+1)(\mathcal{A}_2+1)}\,\{\mathcal{A}_3,\{\mathcal{A}_1,\mathcal{A}_2\}\,\} = (-)^{s_{32}}\big(\mathcal{A}_3\circ(\mathcal{A}_1\circ\mathcal{A}_2)_{cyc}\big)_{cyc}\,, \tag{2.25}$$

where the sign factor s_{13} is given by

$$s_{32} = d_3d_2 + m_3m_2 + (d_1m_3 + d_2m_1 + d_3m_2)\,.$$

We wish to verify that surfaces with the same sewing structure in (2.24) and (2.25) appear with opposite signs. Given three surfaces Σ_1, Σ_2, and Σ_3 belonging to $\mathcal{A}_1, \mathcal{A}_2$, and $\mathcal{A}_3$ respectively, we can use the associativity of the $\circ$-product, and equation (2.17) to write

$$\begin{aligned}\Sigma_1\circ(\Sigma_2\circ\Sigma_3) &= (\Sigma_1\circ\Sigma_2)\circ\Sigma_3\,,\\ &= (C^{-1})^{m_3-1}\big(C^{-1}\Sigma_3\circ C(\Sigma_1\circ\Sigma_2)\big)\,.\end{aligned} \tag{2.26}$$

This equation implies that configurations in the right hand side of (2.25) having the same structure as those appearing in the right hand side of (2.24) have an additional sign factor

$$(m_3-1)(m_1+m_2+m_3-1) + (m_3-1) + (m_1+m_2+1) = m_3m_1+m_3m_2+1 \pmod 2\,.$$

We then confirm that

$$s_{32} + d_3(d_1+d_2) + m_3m_1+m_3m_2+1 = s_{13}+1\,,$$

thus completing the verification that identical surfaces appearing in the various terms of the Jacobi identity cancel out as desired.

Another important property of the antibracket must be its behavior under the action of the boundary operator ∂. For our multiplication we have

$$\partial(\mathcal{A}_1\circ\mathcal{A}_2)_{cyc} = (\partial\mathcal{A}_1\circ\mathcal{A}_2)_{cyc} + (-)^{d_1}(\mathcal{A}_1\circ\partial\mathcal{A}_2)_{cyc}\,, \tag{2.27}$$

and making use of (2.22) the above equation reduces to

$$\partial\{\mathcal{A}_1,\mathcal{A}_2\}_o = \{\partial\mathcal{A}_1,\mathcal{A}_2\}_o + (-)^{\mathcal{A}_1+1}\{\mathcal{A}_1,\partial\mathcal{A}_2\}_o\,, \tag{2.28}$$

which is the expected result.

2.7. Multiple boundary components

We now sketch briefly how the above results should be extended when the complex $\mathcal{P}$ contains moduli spaces $\mathcal{A}$ whose surfaces have more than one boundary component. In this case the moduli space $\mathcal{A}$ will have to satisfy further exchange properties. Let b denote the number of boundary components in each of the surfaces contained in $\mathcal{A}$. The boundaries must be labelled, and Γ_k, with k running from one up to b, will denote the k-th boundary component. We also let m_k denote the number of open string punctures on the boundary component Γ_k. The open string punctures must be labelled cyclically on each boundary component. We denote by C_k the operator that generates a cyclic permutation of the punctures in Γ_k. The moduli space $\mathcal{A}$ must satisfy the cyclicity condition (2.15) for every boundary component: $C_k\mathcal{A} = (-)^{m_k-1}\mathcal{A}$, $k = 1, \cdots, b$. Let now E_{ij}, for $i \neq j$ denote the operator that exchanges the labels of the boundary components Γ_i and Γ_j. A moduli space $\mathcal{A}$ will belong to $\mathcal{P}$, if in addition to the previous constraints it satisfies

$$E_{ij}\,\mathcal{A} = (-)^{m_i+1)(m_j+1)}\,\mathcal{A}\ . \tag{2.29}$$

If a moduli space does not satisfy the above equation, one can always construct a moduli space that does by adding together, with appropriate sign factors, the $b\,!$ inequivalent labellings of the boundaries.

The multiplication "$\circ$" of surfaces is modified in a simple way preserving associativity. This time $\Sigma_1 \circ \Sigma_2$ denotes the sewing of the last puncture of the last boundary component of Σ_1 (the component $\Gamma^{(1)}_{b_1}$, where the superscript refers to the surface) to the first puncture of the first boundary component of Σ_2.(the component $\Gamma^{(2)}_1$). The punctures in the now common boundary component are labelled as we did before. This boundary component is now the component $\Gamma^{(12)}_{b_1}$ of the sewn surface. For $k < b_1$ we define $\Gamma^{(12)}_k = \Gamma^{(1)}_k$ and for $b_1 < k < b_1 + b_2 - 1$ we let $\Gamma^{(12)}_k = \Gamma^{(2)}_{1+k-b_1}$. In words, the labels of the untouched boundaries in Σ_1 are used for the sewn surface, and the boundary components arising from the remaining boundaries on Σ_2 are labelled in ascending order.

Once more we need to quantify the failure of commutativity. Let R denote the operator that reverses the labels of all the boundaries namely $R : \Gamma_k \to \Gamma_{b+1-k}$. The first boundary becomes the last, the last becomes the first, and so on. We now claim that (2.17) gets generalized into the following expression

$$\Sigma_1 \circ \Sigma_2 = R\,(C^{-1}_{b_2})^{m^{(2)}_1 - 1}\big(RC^{-1}_1\Sigma_2 \circ RC_{b_1}\Sigma_1)\,. \tag{2.30}$$

Note that the C operators act on particular boundary components, as indicated by the subscripts. The operators R are necessary inside the parenthesis in order to reverse last and first punctures. The operator R outside the parenthesis achieves an ordering of the punctures in the sewn surface that coincides with that of the left hand side.

Given two moduli spaces $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{P}$ we define

$$(\mathcal{A}_1 \circ \mathcal{A}_2)_* \in \mathcal{P} \tag{2.31}$$

as the space obtained by doing the product, adding all the cyclic permutations on the boundary component where the sewing took place, and then adding with appropriate

signs, copies of the space with boundary labels exchanged such that the total spaces satisfies (2.29). Since both $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfy (2.29) it is sufficient to add over all possible labels for the boundary that was sewn, and once this label is chosen, we sum over splittings of the remaining labels into two groups, one for the left over boundary components of $\mathcal{A}_1$ and the other for the left over boundary components of $\mathcal{A}_2$. We now claim that (2.30) implies that

$$(\mathcal{A}_1 \circ \mathcal{A}_2)_* = (-)^{s_{12}} (\mathcal{A}_1 \circ \mathcal{A}_2)_* \,, \tag{2.32}$$

where the sign factor is given by

$$s_{12} = d_1 d_2 + [1 \, + (1 + b_1 + m^{(1)})(1 + b_2 + m^{(2)})] \,, \tag{2.33}$$

where the first term in the sign factor originates from the orientation, the second term originates from the action of the C and R operators indicated in (2.30). Here $m^{(1)}$ and $m^{(2)}$ denote the total number of punctures in the surfaces belonging to $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively. It is now possible to define the antibracket

$$\{\mathcal{A}_1, \mathcal{A}_2\}_o \equiv (-)^{(1+b_1+m^{(1)})} (\mathcal{A}_1 \circ \mathcal{A}_2)_* \,, \tag{2.34}$$

and we readily verify that

$$\{\mathcal{A}_1, \mathcal{A}_2\}_o = -(-)^{(d_1+1+b_1+m^{(1)})(d_2+1+b_2+m^{(2)})} \{\mathcal{A}_2, \mathcal{A}_1\}_o \,. \tag{2.35}$$

This is exactly the desired result, on account of (2.8) and (2.12).

3. Conditions on the string vertex $\mathcal{V}$

In this section we set up and discuss the recursion relations for the string vertices. They take the form of the BV equation for a string vertex $\mathcal{V}$ representing the formal sum of all string vertices. We use this equation to derive the $\hbar$ and κ dependence of interactions.

We single out from $\mathcal{V}$ special subfamilies of string vertices having simple recursion relations. Those are:

(i) the vertices corresponding to surfaces without boundaries,

(ii) vertices coupling all numbers of open strings on a disk,

(iii) vertices coupling all numbers of open strings to $n \leq N$ closed strings via a disk, together with the vertices coupling $n \leq N+1$ closed strings on a sphere, and

(iv) vertices coupling all numbers of open and closed strings via a disk, together with vertices coupling all numbers of closed strings via a sphere.

3.1. Master equation for String Vertices

As in the closed string case we now consider a string vertex $\mathcal{V}$ defined as a formal sum of string vertices associated to the various moduli spaces

$$\mathcal{V} \equiv \sum_{g,n,b,m} \hbar^p \, \kappa^q \sum_{m_1,\cdots m_b} \mathcal{V}^{g,n}_{b,m} \, . \tag{3.1}$$

All the moduli spaces $\mathcal{V}^{g,n}_{b,m}$ listed above are spaces of degree zero (see (2.7)), and therefore they have the same dimension as $\mathcal{M}^{g,n}_{b,m}$. Here the power p of $\hbar$ and the power q of the string coupling associated to each string vertex are to be determined as functions of g, n, b, and m. As the above expression indicates, having chosen g, n, b and m, one must still sum over the inequivalent ways of splitting the m (labelled) open string punctures over the b boundary components.

There are various moduli spaces that will not be included in the above sum

– the spheres ($g = b = 0$) with $n \leq 2$,

– the torus ($g = 1$, $b = 0$) with $n = 0$, and,

– the disks ($b = 1$, $g = 0$) with $m \leq 2$.

The idea now is to examine the recursion relations between the moduli spaces that follow from

$$\partial\mathcal{V} + \tfrac{1}{2}\{\mathcal{V} \, , \mathcal{V}\} + \hbar\Delta\mathcal{V} = 0 \, , \tag{3.2}$$

where we impose this condition having in mind that it will lead to the string action satisfying the BV master equation. We introduced a power of $\hbar$ in (3.1) in order to account for that factor of $\hbar$ appearing in front of the delta operator. The power of the string coupling was introduced to see what are the conditions that (3.2) implies on the string coupling dependence of the interactions.

It is best to discuss the various terms in the above equation by referring to figure 4. In the left hand side we have the boundary of the general vertex defined by (g, n, b, m). In the right hand side we have five type of configurations. These are, in fact, the same configurations described in section 2.2. The first three configurations correspond to sewing of open string punctures, and the last two configurations correspond to sewing of closed string punctures, as familiar in the closed string case. The configurations involving two vertices arise from the antibracket, and the configurations involving a single vertex arise from the delta operation. Let us examine each in turn.

(i) The first configuration, arising from the antibracket in the open string sector, shows the gluing of open strings lying on boundary component of two different vertices. In this configuration the two boundary components merge to become one, and therefore the total number b of boundaries in the glued surface is $b = b_1 + b_2 - 1$. In addition, $m = m_1 + m_2 - 2$.

(ii) The second configuration arises from the delta operation. Here the two open strings to be glued are in the same boundary component of the string vertex. This operation increases the number of boundary components by one since the gluing splits

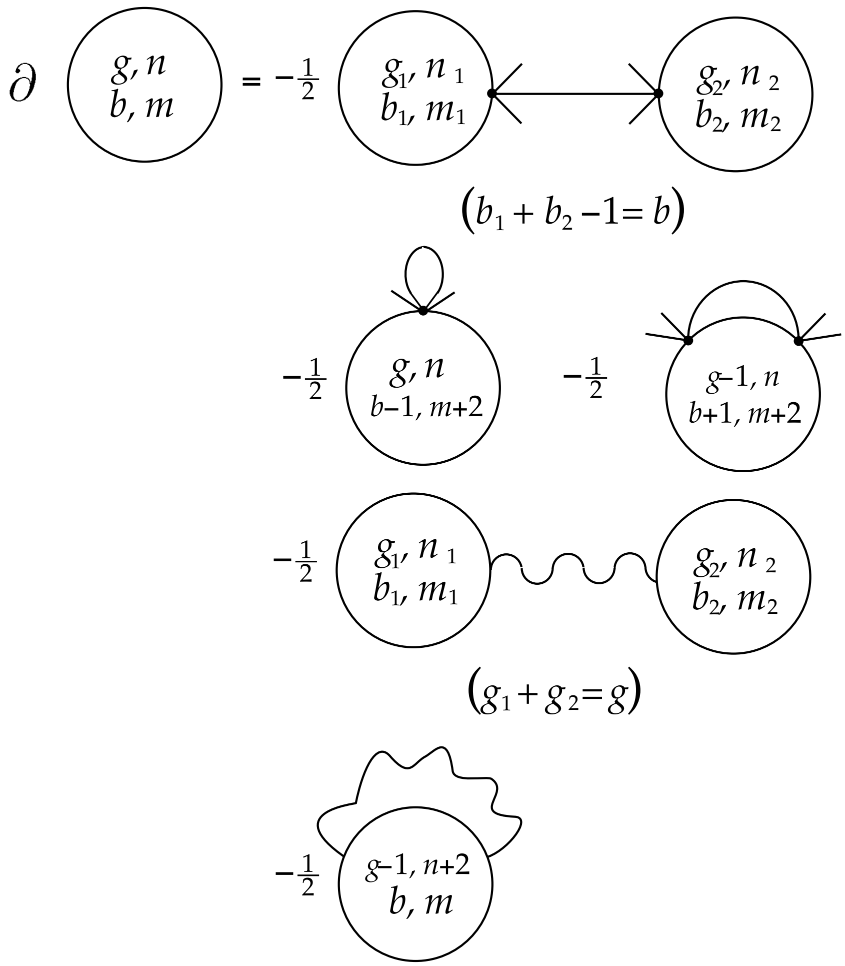


Figure 4. The master equation for open closed string vertices. In the left hand side we have the boundary of the region of moduli space covered by a generic vertex. In the right hand side we have the five inequivalent configurations featuring a single collapsed propagator. The first three configurations arise from sewing of open string punctures, and the last two configurations arise from sewing of closed string punctures.

the boundary in question into two disconnected components. The number of open string punctures is decreased by two units. The relevant blob indicated in the figure must therefore be of type $(g, b-1, n, m+2)$.

(iii) The third configuration also arises from the delta operation. Here the two open strings to be glued are in different boundary components of the string vertex. Just as in the case of configuration (i) the two boundaries involved become a single one. As explained in section 2.2, the genus increases by one. Thus the relevant blob for this case must be of type $(g-1, n, b+1, m+2)$.

(iv) The fourth configuration arises from the antibracket in the closed string sector. As is familiar by now this configuration requires that the genera add to $g = g_1 + g_2$. The closed string punctures must satisfy $n = n_1 + n_2 - 2$, the boundary components must add to $b = b_1 + b_2$ and the open string punctures must add to $m = m_1 + m_2$.

(v) The fifth and final configuration arises from delta in the closed string sector. Here

the genus increases by one unit, the number of closed string punctures is reduced by two units and both the number of boundaries and the number of open string punctures are unchanged.

It is straightforward to verify that all the terms in the geometrical equation are of the same dimensionality. This was guaranteed by our construction. Since the antibracket, delta, and the boundary operator have all degree one, all moduli spaces appearing in (3.2) are guaranteed to have dimension one less than that of the corresponding moduli space $\mathcal{M}$. It follows that moduli spaces in (3.2) of the same type (same g, n, b, m) must have the same dimension.

3.2. Calculation of p and q

We will now see that the values of p and q appearing in (3.1) are severely constrained by the master equation (3.2). If we demand that the three open string vertex appear in the action at zero-th order of $\hbar$ and multiplied by a single power of the coupling constant κ this will fix completely the values of p and q for all interactions. In order to see this clearly it is useful to consider first the sewing properties of the Euler number χ associated to a surface of type (g, n, b, m). We have

$$\chi\,(\Sigma^{g,n}_{b,m}\,) = 2 - 2g - n - b - \tfrac{1}{2}m\,. \tag{3.3}$$

This formula deserves some comment. The closed string punctures here are treated simply as boundary component (note that n appears in the same way as b). Each open string puncture is treated as half of a closed puncture, this is reasonable on account that upon doubling the open string puncture would become a closed string puncture.

There are two important properties to χ. It is additive under sewing of two surfaces

$$\chi(\Sigma_1 \cup \Sigma_2) = \chi(\Sigma_1) + \chi(\Sigma_2)\,. \tag{3.4}$$

This is readily verified by checking the only two possible cases; sewing of closed string punctures, and sewing of open string punctures. It is moreover invariant under sewing of two punctures on the same surface

$$\chi(\cup\Sigma) = \chi(\Sigma)\,, \tag{3.5}$$

and in this case there are the three cases to consider; sewing of two closed string punctures, and sewing of two open string punctures, on the same or on different boundary components. Note that the configurations relevant to the two equations above all appeared in the recursion relations (Fig.4) and were discussed earlier. It is not hard to show that conditions (3.4) and (3.5) fix the function χ to take the form quoted in (3.3) up to an overall multiplicative constant.

Given a moduli space $\mathcal{A}^{g,n}_{b,m}$ of surfaces of type (g, n, b, m) we will define $\overline{\chi}(\mathcal{A})$ to be given by the Euler number χ of the surfaces composing the moduli space ($\overline{\chi}$ is *not* the Euler number of $\mathcal{A}$). We then have that the last two equations imply that

$$\begin{aligned} \overline{\chi}\,(\{A_1, \mathcal{A}_2\}) &= \overline{\chi}\,(\mathcal{A}_1) + \overline{\chi}\,(\mathcal{A}_1)\,, \\ \overline{\chi}\,(\Delta\mathcal{A}) &= \overline{\chi}\,(\mathcal{A})\,. \end{aligned} \tag{3.6}$$

We can now go back to discuss the problem of finding p and q. Let us begin with q, the power of the string coupling. Assume we study the subsector of the master equation

having to do with moduli spaces of some fixed type (g, n, b, m). Such moduli space appears in $\partial\mathcal{V}$, and appears built through an antibracket in $\{\mathcal{V}, \mathcal{V}\}$ and by Δ action in $\Delta\mathcal{V}$. Since the string coupling appears nowhere in the master equation, all such terms must have the same factor κ^q, and therefore, it is necessary that the assignment of q to a moduli space be additive with respect to the antibracket, and invariant under Δ, namely

$$\begin{aligned} q\left(\{\mathcal{A}_1, \mathcal{A}_2\}\right) &= q\left(\mathcal{A}_1\right) + q\left(\mathcal{A}_1\right), \\ q\left(\Delta\mathcal{A}\right) &= q\left(\mathcal{A}\right). \end{aligned} \tag{3.7}$$

These are exactly the same as (3.6) and therefore the most general solution is $q = C\overline{\chi}$, where C is a constant to be determined. We require that q be equal to one for the disk with three open string punctures. For this surface $\overline{\chi} = -1/2$, and therefore we find

$$q = -2\,\overline{\chi} = 4g + 2n + 2b + m - 4\,. \tag{3.8}$$

Let us now discuss the power p of $\hbar$. The master equation has an explicit factor of $\hbar$ in the Δ term, therefore this time we must require that

$$\begin{aligned} p\left(\{A_1, \mathcal{A}_2\}\right) &= p\left(\mathcal{A}_1\right) + p\left(\mathcal{A}_1\right), \\ p\left(\Delta\mathcal{A}\right) &= p\left(\mathcal{A}\right)\, +\, 1\,. \end{aligned} \tag{3.9}$$

It is clear that if we find any solution p, then we can construct many solutions as $p + C\,\overline{\chi}$. In fact, one can show explicitly that this is the most general solution. Therefore, it is enough to find a solution of (3.9). Define

$$\overline{p}(\mathcal{A}) = 1 - \tfrac{1}{2}(n+m)\,, \tag{3.10}$$

for a moduli space of type (g, n, b, m). It is clear that $\overline{p}$ solves (3.9). It then follows that the most general solution is of the type $p = \overline{p} + C\chi$. We require that $p = 0$ for the disk with three open string punctures. This fixes

$$p = -\overline{\chi} + \overline{p} = 2g + \tfrac{1}{2}n + b - 1\,. \tag{3.11}$$

We have therefore found that the string vertex in (3.1) is of the form

$$\mathcal{V} \equiv \sum_{g,n,b,m} \hbar^{-\overline{\chi}+\overline{p}}\,\kappa^{-2\overline{\chi}} \sum_{m_1,\cdots m_b} \mathcal{V}^{g,n}_{b,m}\,. \tag{3.12}$$

Since the string action, except for the kinetic terms will just be $f(\mathcal{V})$, where f is a map to the functions on the total state space of the conformal theory, the above equation gives the order of $\hbar$ for all the interaction terms in the string action. The kinetic terms in the string action, both for the open and closed string sectors appear at $\hbar^0$. Even though they should not be thought as vertices, this is the value of p that follows from equation (3.11) both for the case of a disk with two open string punctures and a sphere with two closed string punctures. Since the classical open string vertices appear at order $\hbar^0$ we include the open string BRST kinetic term at order $\hbar^0$ as well. It is then natural to have the closed string BRST kinetic term appear at the same order of $\hbar$ since it is only the total BRST operator in the conformal theory that defines the boundary operator ∂ at the geometrical level.⋆

⋆ It is of course possible to alter the powers of $\hbar$ by $\hbar$ dependent scalings of the string fields.

We now list the moduli spaces that appear for the first few orders of $\hbar$.

(i) $\hbar^0$: the disk ($b=1$) with $m \geq 3$ open string punctures.

(ii) $\hbar^{1/2}$: the sphere with three closed string punctures, and the disk with one closed string puncture and $m \geq 0$ open string punctures.

(iii) $\hbar^1$: the sphere with four closed string punctures, the torus without punctures, the disk with two closed string punctures and $m \geq 0$ open string punctures, and the annulus with $m \geq 0$ open string punctures split in all possible ways on the two boundary components.

(iv) $\hbar^{3/2}$: the sphere with five closed string punctures, the torus with one closed string puncture, the disk with three closed string punctures and $m \geq 0$ open string punctures, and the annulus with one closed string puncture and $m \geq 0$ open string punctures split in all possible ways on the two boundary components.

Note that one can redefine the string fields and eliminate a separate dependence on $\hbar$ and κ. Indeed it follows from (3.12) that (schematically)

$$\begin{aligned} \frac{1}{\hbar}\mathcal{V} &\sim \sum \hbar^{-\overline{\chi}+\overline{p}-1}\,\kappa^{-2\overline{\chi}}\,\mathcal{V}^{g,n}_{b,m} \\ &\sim \sum (\hbar\kappa^2)^{-\overline{\chi}+\overline{p}-1}\left[\kappa^{n+m}\,\mathcal{V}^{g,n}_{b,m}\right], \end{aligned} \tag{3.13}$$

where use was made of (3.10). It follows from the above equation that upon a simple coupling constant rescaling the terms in brackets can be thought as the new string vertices, and $S/\hbar$ only depends on $\hbar\kappa^2$.

3.3. Sub-recursion relations

The recursion relations (3.2) relate all of the string vertices of the open-closed string theory. All relevant string vertices appear in $\mathcal{V}$. It is of interest to isolate sub-families of string vertices related by simpler recursion relations.

Closed string vertices Closed string vertices, namely, surfaces with no boundaries define a subfamily because it is not possible to obtain surfaces without boundaries by sewing operations on surfaces with boundaries (sewing of open string punctures glues only pieces of boundary components). If we define

$$\mathcal{V}^c = \sum_{g,n} \hbar^p \kappa^q \mathcal{V}_{g,n}\,, \tag{3.14}$$

we get the recursion relation

$$\partial\mathcal{V}^c + \tfrac{1}{2}\{\mathcal{V}^c\,,\mathcal{V}^c\} + \hbar\Delta\mathcal{V}^c = 0\,, \tag{3.15}$$

for the closed string sub-family. The values of p and q need not be those given earlier, since those were determined by conditions on the three open string vertex. If one deals with closed strings only one requires that $p=0$ and $q=1$ for the three punctured sphere, giving $p=g$, $q=-\overline{\chi}=2g+n-2$. Note that the classical closed string vertices, defined by only summing over $g=0$ in the above, are also a subfamily of vertices.

Disks with open string punctures These are the vertices that define classical open string field theory. We define

$$\mathcal{V}_0 = \sum_{m \geq 3} \mathcal{V}^{0,0}_{1,m}\,, \tag{3.16}$$

dropping, for convenience all factors of $\hbar$ and κ. The zero subscript denotes zero number of closed strings. We must explain why this is a subfamily. The idea is simple, surfaces outside this family cannot produce surfaces in the family by sewing operations. This can be established at the same time as we find the recursion relation satisfied by $\mathcal{V}_0$ by inspection of Fig. 4. Here we argue that only the first term in the right hand side is relevant. The second configuration is not relevant because in order to get a surface with one boundary component we would need a surface with $b = 0$ and then open string sewing is impossible. The third configuration is also impossible since we would need a string vertex with $g = -1$. The fourth configuration, having closed string tree-like sewing would require that the boundary component be in one of the vertices only, the other vertex must be a pure closed string vertex. Since closed string vertices start with $n \geq 3$ we would have two external closed string punctures, in contradiction with the fact that we must just get a disk open string punctures. The fifth configuration is irrelevant because of genus. Therefore, we have shown that

$$\partial \mathcal{V}_0 + \tfrac{1}{2}\{\mathcal{V}_0\,, \mathcal{V}_0\} = 0\,. \tag{3.17}$$

The antibracket here is the complete antibracket, but given the absence of closed string punctures it just acts on the open string sector. Note that the number of open string punctures in $\mathcal{V}_0$ goes, in general, from three to infinity. Thus, in general we have a fully nonpolynomial classical open string field theory. In algebraic terms, the structure governing this theory is an A_∞ algebra.

Disks with open string punctures and one or zero closed strings We now introduce the set of vertices described by disks with one closed string puncture, and $m \geq 0$ open string punctures. We define

$$\mathcal{V}_1 = \sum_{m \geq 0} \mathcal{V}^{0,1}_{1,m}\,, \tag{3.18}$$

We now claim that there are recursion relations involving this family $\mathcal{V}_1$ and the family $\mathcal{V}_0$. Again, inspecting Fig. 4 and by essentially identical arguments, we argue that only the first term in the right hand side is relevant. It then follows that

$$\partial \mathcal{V}_1 + \{\mathcal{V}_0\,, \mathcal{V}_1\} = 0\,. \tag{3.19}$$

Disks with open string punctures and two or less closed strings We now introduce the set of vertices described by disks with two closed string punctures, and $m \geq 0$ open string punctures We define

$$\mathcal{V}_2 = \sum_{m \geq 0} \mathcal{V}^{0,2}_{1,m}\,, \tag{3.20}$$

This time, inspection of Fig. 4, reveals that, in addition to the first configuration, the vertices $\mathcal{V}_1$ can now be sewn to the three string vertex V^c_3 to give a surface with one

boundary and two closed string punctures. We therefore have t then follows that

$$\partial\mathcal{V}_2 + \{\mathcal{V}_0, \mathcal{V}_2\} + \tfrac{1}{2}\{\mathcal{V}_1, \mathcal{V}_1\}_o + \{\mathcal{V}_1, \mathcal{V}_3^c\} = 0\,. \tag{3.21}$$

It is fairly clear that we could continue in this fashion adding closed string punctures one at a time. If we define $\mathcal{V}_N$ as disks with N closed string punctures, their recursion relations will involve the other $\mathcal{V}_n$'s with $n < N$, and all classical closed string vertices up to $\mathcal{V}_{N+1}^c$. Since we will not have any explicit use for the particular families $\mathcal{V}_{N>2}$, we now consider all of the families $\mathcal{V}_N$ put together.

<u>Disks with all numbers of open and closed string punctures</u> We now introduce the set of vertices $\overline{\mathcal{V}}$ that includes the families $\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2$ introduced earlier along with all their higher counterparts.

$$\overline{\mathcal{V}} = \sum_{n\geq 0} \mathcal{V}_n\,, \tag{3.22}$$

This time, inspection of Fig. 4, reveals that,

$$\partial\overline{\mathcal{V}} + \tfrac{1}{2}\{\overline{\mathcal{V}}, \overline{\mathcal{V}}\}_o + \{\overline{\mathcal{V}}, \mathcal{V}^c\} = 0\,. \tag{3.23}$$

Note that the second term is the antibracket on the open string sector only. Had we used the closed sting antibracket we would get surfaces with two boundary components.

There are other subfamilies, but we have found no explicit use for them. A curious one is the family $\widetilde{\mathcal{V}}$ that includes the set of all genus zero vertices having all numbers of boundary components and all numbers of open and closed string punctures. The vertex $\widetilde{\mathcal{V}}$ satisfies an equation of the type

$$\partial\widetilde{\mathcal{V}} + \tfrac{1}{2}\{\widetilde{\mathcal{V}}, \widetilde{\mathcal{V}}\} + \Delta_o'\widetilde{\mathcal{V}} = 0\,, \tag{3.24}$$

where Δ_o' denotes the open string delta restricted to the case when it acts on two punctures that are on the same boundary component. This must be the case since the operation of sewing two open string punctures on different boundary components of a surface increases its genus.

4. Minimal area string diagrams and low dimension vertices

In order to write a open-closed string field theory we have to determine the open-closed string vertices. In other words we have to find $\mathcal{V}$. This involves specifying, for each moduli space included in (3.1), the region of the moduli space that must be thought as a string vertex. Throughout this region we must know how to specify local coordinates on every puncture.

The purpose of the present section is to show explicitly how to do this for the relevant low dimension moduli spaces, and to explain how the procedure can be carried out in general. The strategy was summarized in Ref.[17]; one defines the string diagram associated to every surface, and uses this string diagram to decide whether or not the

surface in question belongs to the string vertex. A string diagram corresponding to a punctured Riemann surface R, is the surface R equipped with local complex coordinates at the punctures. Since suitable (Weyl) metrics on Riemann surfaces can be used to define local coordinates around the punctures [18,19], the string diagram for the surface R can be thought as R equipped with a suitable metric ρ.

For open-closed string theory we propose again to use minimal area metrics, this time requiring that open curves be greater than π (to get factorization in open string channels) and that closed curves be longer than 2π (in order to get factorization in closed string channels).

> **Minimal Area Problem for Open-Closed String Theory**: *Given a genus g Riemann surface R with b boundaries, m punctures on the boundaries and n punctures in the interior ($g, b, n, m \geq 0$) the string diagram is defined by the metric ρ of minimal (reduced) area under the condition that the length of any nontrivial open curve in R, with endpoints at the boundaries be greater or equal to π and that the length of any nontrivial closed curve be greater or equal to 2π.*

As in our earlier work, since the area is infinite when there are punctures, one must use the reduced area, which is a regularized area obtained by subtracting the leading logarithmic divergence [20]. All relevant properties of area hold for the reduced area. Open curves are curves whose endpoints lie on boundary components.

This minimal area problem produces suitable metrics. By "suitable" we have two properties in mind. First, it allows us to define local coordinates at the punctures. This is because the metric in the neighborhood of a closed string puncture is that of a flat cylinder of circumference 2π, and the metric in the neighborhood of an open string puncture is that of a flat strip of width 2π. Both the cylinder and the strips have well defined ends (where they meet the rest of the surface) and thus define maximal cylinders (disks) and maximal strips (half-disks). These are helpful to define the local coordinates around the punctures, as will be explained shortly.

The second property is that gluing of surfaces equipped with minimal area metrics must give surfaces equipped with minimal area metrics. This requires that the definition of local coordinates be done with some care in order that the sewing procedure will not introduce short nontrival curves. The maximal domains around the punctures cannot be used. Somewhat smaller canonical domains will be used. The canonical closed string cylinder (or disk) is defined to be bounded by the closed geodesic located a distance π from the end of the maximal cylinder. If we remove the canonical cylinder we are leaving a "stub" of length π attached to the surface. The canonical strip (or half-disk) is defined to be bounded by the open geodesic located a distance π from the end of the maximal strip. If we remove the canonical strip we are thus leaving a "strip" of length π attached to the surface (see Fig. 5). Upon sewing, the left-over stubs and short strips ensure that we cannot generate nontrivial closed curves shorter than 2π nor nontrivial open curves shorter than π.

The simplest string diagram is the open-closed string transition corresponding to a disk with one closed string puncture and one open string puncture. This string diagram

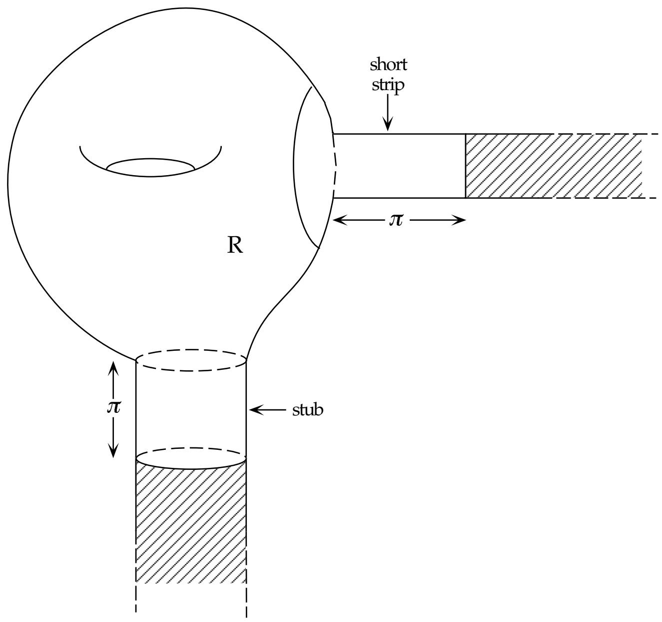
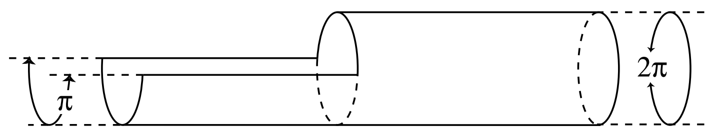


Figure 5. A surface R equipped with a minimal area metric. The boundaries of the maximal cylinder and of the maximal strip are indicated as dotted lines. The canonical cylinder and the canonical strip are shown dashed. They do not coincide with the maximal cylinder and the maximal disk, but rather differ by a length π cylindrical "stub" attached to the surface, and a length π short "strip" attached to the surface.

is illustrated in Fig. 6(a). It is very different from the one used in the light-cone, where an open string just closes up when the two endpoints get close to each other. Here an open string of length π travels until all of the sudden an extra segment of length π appears and makes up a closed string of length 2π. It is simple to prove that this is a minimal area metric. Imagine cutting the surface at the open string geodesic where the open string turns into a closed string. This gives us a a semiinfinite strip and a semiinfinite cylinder. If there is a metric with less area on this surface, it must have less area in at least one of these two pieces. But this is impossible since flat cylinders and flat strips already minimize area under the length conditions.

The open string theory of Witten [2] is related to a different minimal area problem. Its diagrams use the metric of minimal area under the condition that nontrivial open curves with boundary endpoints be at least of length π [21,22]. This minimal area problem allows for short nontrivial closed curves, in contrast with the open-closed problem discussed above. In order to incorporate closed string punctures into Witten's formulation one simply declares that the open curves in the minimal area problem cannot be moved across closed string punctures [23]. The resulting string diagrams are not manifestly factorizable in the closed string channels.

<u>The specification of the string vertex</u> In general, the minimal area problem determines the string vertex $\mathcal{V}$ as follows. Given a surface R, we find its minimal area metric. Then $R \in \mathcal{V}$ (R belongs to the string vertex) if there is no internal propagator in the string diagram. Namely, if we cannot find an internal flat cylinder or an internal flat strip of length greater than or equal to 2π. The logic is simple [17]. Since string vertices have



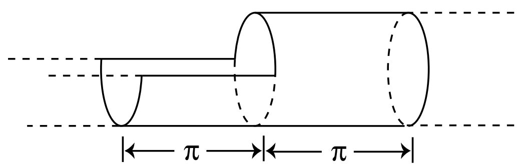


Figure 6. (a)The open-closed minimal area string diagram. An open string of length π suddenly becomes a closed string of length 2π. (b) The corresponding string vertex with a stub and a short strip.

stubs and short strips, whenever we form a Feynman graph by sewing we must generate either cylinders or strips longer than 2π. If such cylinders or strips cannot be found in a string diagram, the string diagram in question cannot have arisen from a Feynman graph, and must therefore be included in the string vertex.

4.1. Classical open string theories

For surfaces corresponding to open string tree amplitudes (disks with punctures on the boundary), both the open-closed minimal area problem of this paper, and the pure open minimal area problem alluded above yield the same minimal area metrics. This is because such surfaces have no nontrivial closed curves. For the case of loop amplitudes the two minimal area problems give different metrics.

Even though the metrics are the same for the case of disks with open string punctures, the string vertices *are not the same*. In [2] there is only a three open string vertex corresponding to the disk with three open string punctures. In classical open-closed string field theory there will be open string vertices for corresponding to disks with m open string punctures, for all values of $m \geq 3$. This is due to the inclusion of length π open strips on the legs of the string vertices. The higher classical string vertices can be obtained as the string diagrams built using Witten type three-string vertices and open string propagators shorter than 2π.

If we are only concerned about the classical theory of open strings the open strips attached to the vertices can have any arbitrary length l. The family of vertices $\mathcal{V}_0(l)$ introduced earlier would depend on the parameter l. For $l = 0$ we find the Witten theory, including only a three punctured disk, while for any $l \neq 0$ we get moduli spaces of disks with all numbers of open string punctures. in particular for $l = \pi$ we get the classical open-closed string theory. For all values of l the recursion relations $\partial\mathcal{V}_0(l) + \frac{1}{2}\{\mathcal{V}_0(l), \mathcal{V}_0(l)\} = 0$ hold. It is clear that the parameter l defines a deformation of open string vertices

interpolating from the Witten theory to the open-closed string theory of this paper. As shown by Hata and the author in the context of closed strings [24], deformations of string vertices preserving the recursion relations give rise to string field theories related by canonical transformations of the antibracket. As elaborated by Gaberdiel and the author [15], this means that the family of A_∞ homotopy algebras $\mathbf{a}(l)$ underlying these l-dependent string field theories are all homotopy equivalent in the same vector space.

4.2. Extracting the Dimension Zero Vertices

Out of the dimension zero vertices (listed in §2.1) there are six that do not appear in the string vertex $\mathcal{V}$. These are the zero, one and twice punctured sphere, and the disk with zero, one or two open string punctures. The above two-punctured surfaces are used as building blocks for the symplectic forms. The above once-punctured surfaces would be relevant in formulations around non-conformal backgrounds. The surfaces with no punctures would only introduce constants (see, section 9, however).

Next are the three punctured sphere and the disk with three open punctures. Those are relevant vertices and are taken to correspond to be the symmetric closed string vertex and the symmetric open string vertex [2] since they solve the minimal area problem. For the closed string vertex we must include a stub of length π in each of the three cylinders, and for the open string vertex we must include a short strip of length π on each of the three legs (Fig. 7)

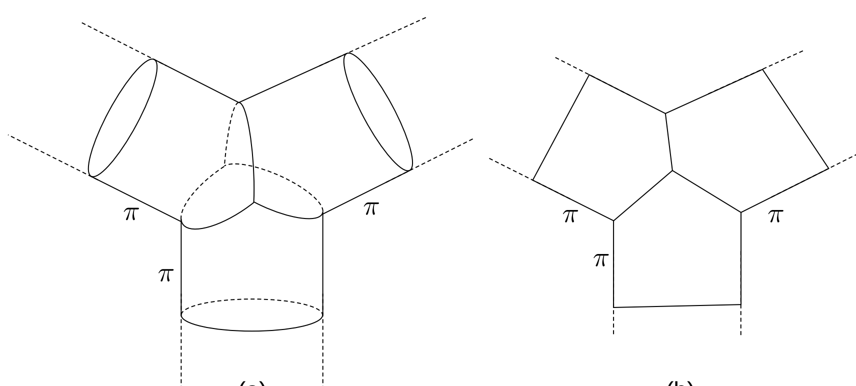


Figure 7. (a) The three closed string vertex with its stubs of length π included. (b) The open string vertex with its short strips of length π included.

Next at dimension zero is the disk with one open and one closed puncture, corresponding to the open-closed string diagram considered before in Figure 6(a). The vertex, shown in Fig. 6(b) includes a stub and a short strip. Finally, we have a disk with one closed string puncture only. This surface represents a closed string that just stops. In our minimal area framework it is a semiinfinite cylinder of circumference 2π (Fig. 8(a)). As a vertex, it looks like a short cylinder with two boundaries, one to be connected to a propagator and the other left open (Fig. 8(b)). This surface has a conformal Killing vector.

(a)

(b)

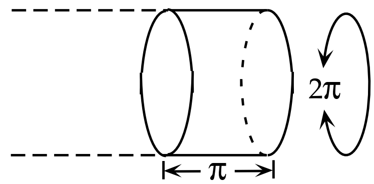


Figure 8. (a) This is the string diagram for a disk with a closed string puncture. It represents an incoming closed string that simply stops. (b) As usual, we must leave a stub of length 2π in the corresponding string vertex.

4.3. Extracting the Dimension One Vertices

As listed in §2.1 there are four moduli spaces that must be considered. We examine each of them to determine both the minimal area string diagrams and to extract the string vertices.

The disk with four open string punctures The boundary of a moduli space of four punctured disks arises from two three-string open vertices sewn together (see Fig. 4). For a given cyclic labelling of the punctures there are four superficially different ways to assign cyclically the labels in the sewn diagram. The four configurations break up into two groups of identical configurations, the factor of one-half in the geometrical equation showing that the boundary of the moduli space in question is given by two sewing configurations, one with a plus sign and one with minus sign (recall the cyclic factor is minus one to the number of punctures minus one). This is the familiar fact that an open string four vertex must interpolate from an s-channel to a t-channel configuration. Since the three string vertex has a short strip of length π attached, the boundaries of the moduli space of four punctured disks are minimal area metrics with internal strips of length 2π. The four string vertex that we need includes all four punctured disks whose minimal area metrics have an internal strip of length less than or equal to 2π. The external legs are amputated leaving strips of length π.

The disk with one closed string puncture and two open string punctures

This moduli space corresponds to an open-open-closed interaction, and will be discussed in detail in section 6.3. Since the discussion in that section will be based on the associative open string vertex, the open-open-closed vertex of the open-closed theory also includes the region of moduli space that arises when the closed-open vertex is joined to the three-open-string vertex, and we let the intermediate strip shrink from the initial length of 2π down to zero.

Disk with two closed string punctures The moduli space can be represented by taking a unit disk with one of the closed strings in the center and the other in the positive horizontal

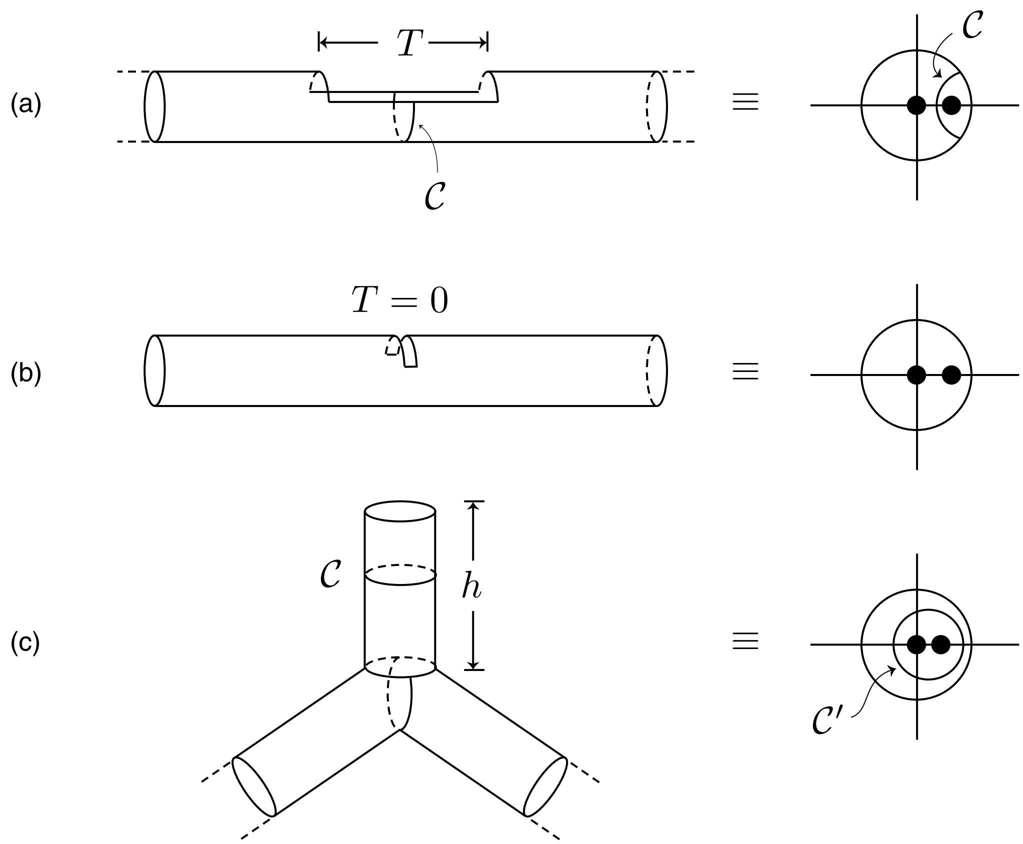


Figure 9. Minimal area string diagrams for a disk with two closed string punctures. (a) A closed string turns into an open string that propagates for time T and then turns back into a closed string. When $T \to \infty$ we get open string poles. (b) The above configuration when $T = 0$. We have a slit on the infinite cylinder. (c) A stub of length h emerges from the slit. As $h \to \infty$ we get closed strings going into the vacuum.

axis. The position of the second puncture is the modular parameter. Consider a closed string turning into an open string and turning back into a closed string, as shown in Fig. 9(a). The intermediate time is denoted by T. It is clear that for any value of T this is a minimal area surface, since its double is a closed string diagram, a four closed string scattering amplitude with an intermediate propagator. When $T \to \infty$ we find open string poles. This is due to the presence of short noncontractible open curves in the disk representation. In this representation one of the punctures is approaching the boundary and the short curve is $\mathcal{C}$ (Fig. 9(a)). When T becomes zero one is left with a cylinder with a slit of perimeter 2π (Fig. 9(b)). But this cannot be the end of the story. There is nothing singular about this surface, and therefore we are missing the region where the two closed string punctures are very close to each other in the disk representation. When this happens a new closed string appears, the noncontractible curve surrounding the two punctures, this closed string is homotopic to the boundary of the disk. Indeed, in the minimal area string diagrams when T becomes zero a cylindrical stub begins to grow out of the slit (Fig. 9(c)). The height h of the stub goes from zero to infinity supplying the remaining region of moduli space. The curve $\mathcal{C}$ represents the new closed string that is

going into the vacuum. It is clear that if the boundary is mapped to the unit disk the punctures are becoming close to each other as $h \to \infty$. This region of moduli space can also be pictured as a two punctured sphere with a boundary that is becoming smaller and smaller. This viewpoint arises if we demand, in the disk representation, that the two external closed string punctures be fixed. Then the boundary must grow until it becomes a tiny circle around infinity. As the boundary shrinks we are obtaining a three punctured sphere. Indeed, as $h \to \infty$, the minimal area string diagram becomes the standard minimal area diagram representing a three punctured sphere.

Clearly, the region $T \in [2\pi, \infty\}$ is produced by the Feynman graph using two open-closed string vertices and an open string propagator. Moreover, the region $h \geq 2\pi$ is produced by a graph containing the three-closed-string vertex, a propagator and the closed-boundary vertex. The missing region $(T \leq 2\pi) \cup (h \leq 2\pi)$ must be supplied by the closed-closed-boundary vertex. As usual, the vertex is defined by the set of surfaces, each having the external strips and cylinders amputated down to length π.

<u>Annulus with one open string puncture</u>

The outer boundary of the annulus is set to be the $|z| = 1$ boundary of the disk, the open string puncture is at $z = 1$, and the modular parameter is chosen to be the radius r of the inner boundary of the annulus (see Fig.10) . When $r \to 1$ one has very short open curves going from one boundary to the other, showing the presence of open string poles. The relevant string diagram in this region is given by the "tadpole" graph shown in Fig. 10(a), where the length T of the intermediate strip is very big, in particular much longer than 2π. This is a solution of the minimal area problem because its double is a closed string diagram, a one loop tadpole with a propagator of length T.

In the Witten formulation of open string theory one lets T go all the way to zero where a singular configuration is encountered (Fig.11). The Fock space representation of this configuration involves divergent quantities, and gives a potential violation of the BV master equation [25].*

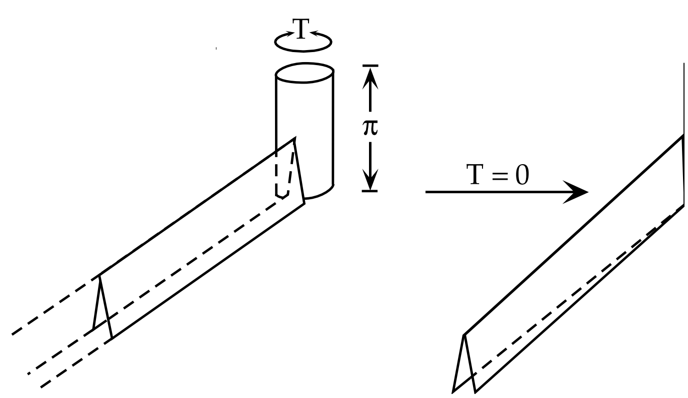


Figure 11. An open string one loop tadpole. The parameter T is the length of the internal propagator. As $T \to 0$ we obtain a singular limit.

∗ An early discussion arguing that the term can be set to zero can be found in [26]. Reference [27] has shown that the BV master equation is satisfied in the midpoint formalism of open string theory. The midpoint formulation of open string theory [18] also requires careful regularization.

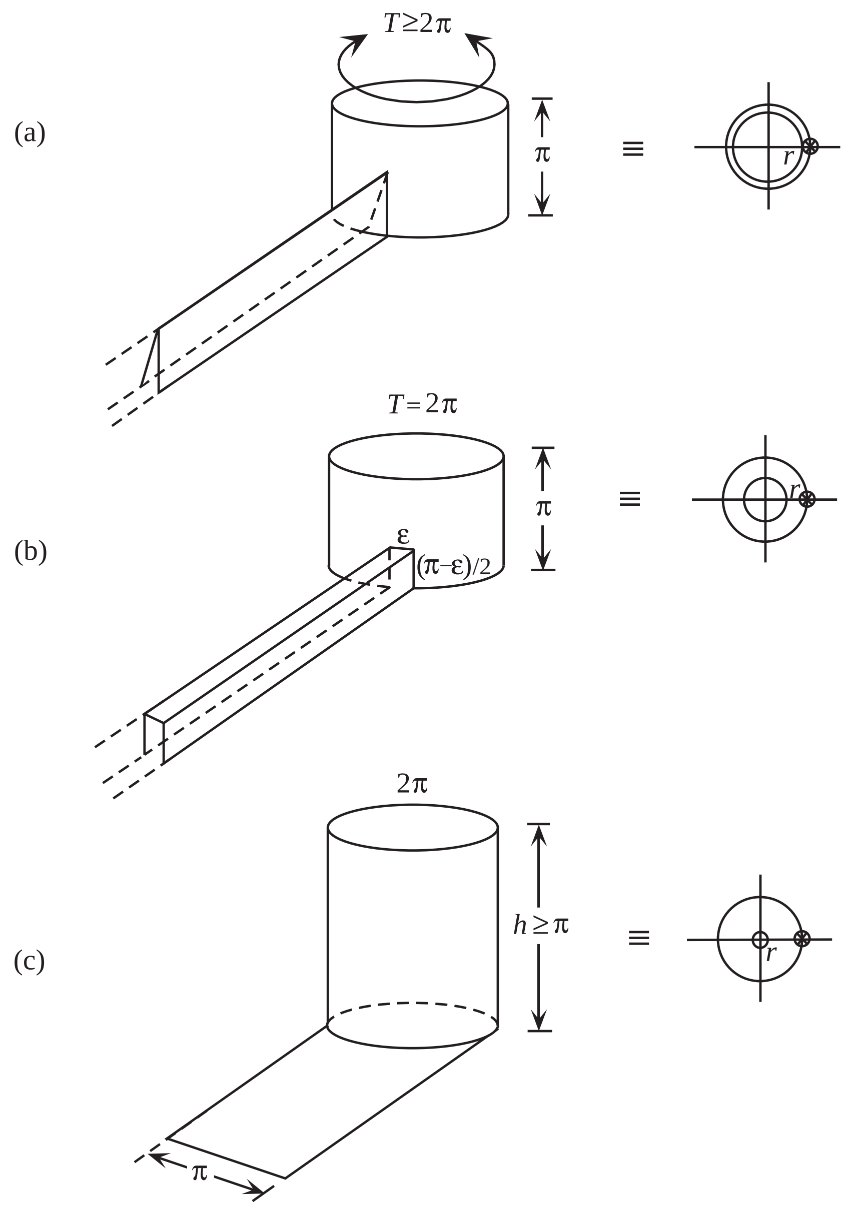


Figure10. Minimal area string diagrams for an annulus with an open string puncture. (a) When $T \to \infty$ we get open string poles. The length T cannot be shorter than 2π, since this would violate the length condition on nontrivial closed curves. (b) When $T = 2\pi$ the slit gets deformed into a rectangular hole of width ϵ and height $(\pi - \epsilon)/2$. When $\epsilon = \pi$ the configuration turns into that of (c), with $h = \pi$. When $h \to \infty$, we get closed strings going into the vacuum.

In open-closed theory, however, we do not run into problems since T cannot shrink below 2π. In fact, due to the strips of length π in the open string vertex, this endpoint is a collapsed tadpole. While we do not know the explicit minimal area metrics, what happens after that must be fairly close to the following: the slit must be deformed into a rectangular hole of width ϵ and height $(\pi - \epsilon)/2$, as shown in Fig. 10(b), with $\epsilon \in [0, \pi]$. When $\epsilon = \pi$ we get a minimal area metric matching smoothly with the $h = \pi$ configuration of Fig. 10(c). Then h begins to grow. When $h = 2\pi$, the string diagram corresponds to

a Feynman graph built with an open-closed vertex and a closed-boundary vertex in the limit when the propagator collapses. This graph provides the remaining part of the moduli space. The region $h \to \infty$ corresponds to closed strings going into the vacuum, or $r \to 0$. The elementary vertex must therefore comprise the region $\{0 \leq \epsilon \leq \pi\} \cup \{\pi \leq h \leq 2\pi\}$.

5. The open-closed master action

The aim of the present section is to describe the construction of the open-closed string field action, and to explain why our earlier work on the string vertices implies that the master equation is satisfied.

The string field action is a function on the full vector space $\mathcal{H}$ of the open-closed theory. This vector space is the direct sum $\mathcal{H} = \mathcal{H}_o \oplus \mathcal{H}_c$ where $\mathcal{H}_o$ denotes the open string state space, and $\mathcal{H}_c$ denotes the closed string state space. As is well known (see, for example 17]) $\mathcal{H}_c$ is defined as the set of states of the matter-ghost CFT that annihilated both by b_0^- and L_0^-. There is no such constraint for $\mathcal{H}_o$, which comprises all states of the boundary ghost-matter CFT.

The complete space $\mathcal{H}$ must be equipped with a invertible odd bilinear form, the symplectic structure of BV quantization. The symplectic structure we introduce is diagonal; it couples vectors in $\mathcal{H}_c$ to vectors in $\mathcal{H}_c$, and vectors in $\mathcal{H}_o$ to vectors in $\mathcal{H}_o$. The bilinear form must also be odd, and therefore must couple odd vectors to even vectors. Recall that in the closed string theory [17] the symplectic form was described as a bra $\langle \omega_{12}^c| \in \mathcal{H}_c^* \otimes \mathcal{H}_c^*$. This bra is used to define $\langle A, B\rangle_c \equiv \langle \omega_{12}^c|A\rangle_1|B\rangle_2$. Given two closed string states, the bilinear form $\langle \cdot, \cdot\rangle_c$ gives the correlator of the two states and a c_0^- ghost insertion on a canonical two-punctured. Since the sum of ghost numbers in a closed string correlator is six, this insertion guarantees that the bilinear form couples vectors of odd ghost number to vectors of even ghost number, and as a consequence odd vectors to even vectors. Moreover, the ghost insertion resulted in the fact that $\langle \omega_{12}^c| = -\langle \omega_{21}^c|$. This property implies that the string field should be Grassmann even in order for the obvious kinetic term $\langle \Psi, Q_c\Psi\rangle$ not to vanish.

For open strings the correlator of two states on a disk requires that the sum of ghost numbers be equal to three. We can therefore naturally obtain an odd symplectic form $\langle \cdot, \cdot\rangle_o$ by defining it to be give correlator of two states, without extra insertions, on the canonical two-punctured disk. As a Riemann surface, the canonical two-punctured disk is symmetric under the exchange of the punctures. We may then be tempted to conclude that for open strings $\langle \omega_{12}^o| = +\langle \omega_{21}^o|$. If that is the case, the naive open string kinetic term $\langle \Phi, Q_o\Phi\rangle$ vanishes unless the string field is odd. This is in fact the choice in the formulation of Ref.[2]. It seems more convenient, however, to have an even open string field, since this allows us to treat the closed and the open string field with the same set of conventions. In the odd string field convention, the moduli spaces entering into the string action would have to have nontrivial degrees according to the number of open strings they couple, rather than degree zero, as we described in an earlier section. The distinction between having an odd or an even string field is at any rate a matter of convention, and we will work with the even string field convention.

In the odd string field convention, the typical classical bosonic open string field $c_1 V|0\rangle$ is odd, and the in-vacuum $|0\rangle$ is declared an even vector.* It follows from $\langle 0|c_{-1}c_o c_1|0\rangle = 1$ that the out-vacuum $\langle 0|$ should be declared odd. In the conventions to be used here we take $|0\rangle$ to be odd and $\langle 0|$ to be even. The string field will be even, and we should expect the symplectic form to be antisymmetric under the exchange of its labels $\langle\omega_{12}^{o}| = -\langle\omega_{21}^{o}|$, as this is required for the canonical kinetic term $\langle\Phi, Q_o\Phi\rangle_o$ not to vanish when the string field is even. We can examine the exchange property explicitly for the case of open bosonic strings. In this case the Fock space representation of the bra is given by [25]

$$
\begin{aligned}
\langle\omega_{12}^{o}| = \int \frac{dp}{(2\pi)^d}\; {}_1\langle p|c_{-1}^{(1)} \;\cdot\; {}_2\langle -p|c_{-1}^{(2)} \;\cdot\; (c_0^{(1)} + c_0^{(2)}) \\
\cdot\; \exp\Big[-\sum_{n=1}^{\infty}(-)^n(\alpha_n^{(1)}\cdot\alpha_n^{(2)} + c_n^{(2)}b_n^{(1)} + c_n^{(1)}b_n^{(2)})\Big]\,.
\end{aligned}
\tag{5.1}
$$

Since in our convention $\langle 0|$ is even, the bra $\langle p|$ is also even, and $\langle p|c_{-1}$ is odd. It follows from examination of the first line of the above equation that the bra $\langle\omega\,^{o}|$ is indeed antisymmetric under the exchange of labels.

The open and closed bilinear forms satisfy

$$
\begin{aligned}
\langle\omega_{12}^{o}|(Q_o^{(1)} + Q_o^{(2)}) = 0\,,\\
\langle\omega_{12}^{c}|(Q_c^{(1)} + Q_c^{(2)}) = 0\,,
\end{aligned}
\tag{5.2}
$$

and both are invertible, with symmetric inverses $|S_{12}^{o}\rangle = |S_{21}^{o}\rangle$ and $|S_{12}^{c}\rangle = |S_{21}^{c}\rangle$ satisfying

$$
\begin{aligned}
\langle\omega_{12}^{o}|S_{23}^{o}\rangle = {}_3\mathbf{1}_1^{o}, \quad (Q_o^{(1)} + Q_o^{(2)})|S_{12}^{o}\rangle = 0\,,\\
\langle\omega_{12}^{c}|S_{23}^{c}\rangle = {}_3\mathbf{1}_1^{c}, \quad (Q_c^{(1)} + Q_c^{(2)})|S_{12}^{c}\rangle = 0\,,
\end{aligned}
\tag{5.3}
$$

The string fields are defined by the usual expansions

$$
|\Phi\rangle = \sum_i |\Phi_i\rangle\phi^i\,, \quad |\Psi\rangle = \sum_i |\Psi_i\rangle\psi^i\,,
$$

where $|\Phi_i\rangle$ and $|\Psi_i\rangle$ are basis vectors for the state spaces $\mathcal{H}_o$ and $\mathcal{H}_c$ respectively, and ϕ^i and ψ^i denote the target space open and closed string fields respectively.

For functions on the state space $\mathcal{H}$ we define the BV antibracket and the delta operator by the expressions

$$
\{A, B\} = \frac{\partial_r A}{\partial\phi^i}\,\omega_o^{ij}\,\frac{\partial_l B}{\partial\phi^j} \;+\; \frac{\partial_r A}{\partial\psi^i}\,\omega_c^{ij}\,\frac{\partial_l B}{\partial\psi^j}\,,
\tag{5.4}
$$

$$
\Delta A = \frac{1}{2}(-)^i\frac{\partial_l}{\partial\phi^i}\Big(\omega_o^{ij}\frac{\partial_l A}{\partial\phi^j}\Big) + \frac{1}{2}(-)^i\frac{\partial_l}{\partial\psi^i}\Big(\omega_c^{ij}\frac{\partial_l A}{\partial\psi^j}\Big)\,,
\tag{5.5}
$$

where the matrices ω_o^{ij} and ω_c^{ij} are the inverses of the matrices ω_{ij}^{o} and ω_{ij}^{c} respectively. The latter are defined from the symplectic forms as $\omega_{ij}^{o} = \langle\Phi_i, \Phi_j\rangle_o$ and $\omega_{ij}^{c} = \langle\Psi_i, \Psi_j\rangle_c$.

$\star$ While our remarks are all made explicitly in the context of bosonic strings, they are also applicable to the case of NS superstrings.

The open-closed string field action S is a function in $\mathcal{H}$, or in other words it is a function of both the open and closed string fields $S(\Phi, \Psi)$. Consistent quantization requires that it should satisfy the BV master equation

$$\tfrac{1}{2}\{S, S\} + \hbar \Delta S = 0\,. \tag{5.6}$$

In order to construct an action that manifestly satisfies this equation we must now consider a map from the complex $\mathcal{P}$ of moduli spaces of bordered surfaces to the space of functions on $\mathcal{H}$. We define

$$f(\mathcal{A}^{g,n}_{b,m}) = \Big[\frac{1}{n!}\frac{1}{b!}\prod_{k=1}^{b}\frac{1}{m_k}\Big] \int_{\mathcal{A}} \langle \Omega | \underbrace{|\Psi\rangle \cdots |\Psi\rangle}_{n} \prod_{k=1}^{b} \underbrace{|\Phi\rangle \cdots |\Phi\rangle}_{m_k} \tag{5.7}$$

where we have included normalization factors $n!$ for the number of closed strings, $b!$ for the number of boundary components, and m_k for each boundary component. In the integrand we insert n copies of the closed string field on the n punctures, and m_k copies of the open string field on the k-th boundary component. The bra $\langle\Omega|$ denotes a form on the moduli space of surfaces, its precise definition for the case of closed strings was given in Refs.[17,6], and no doubt a similar explicit discussion could be done for the open string sector. The main property of the map $f : \mathcal{P} \to C^\infty(\mathcal{H})$ is that it defines a map of BV algebras. In analogy to the closed string case discussed explicitly in [6] we now should have

$$\begin{aligned} f(\{\mathcal{A}, \mathcal{B}\}) &= -\{f(\mathcal{A}), f(\mathcal{B})\}\,, \\ \Delta(f(\mathcal{A})) &= -f(\Delta\mathcal{A}))\,. \end{aligned} \tag{5.8}$$

With a little abuse of notation we will denote by Q_o the function in $\mathcal{H}$ that defines the open string kinetic term, namely $\frac{1}{2}\langle\Phi, Q_o\Phi\rangle_o$, and by Q_c the function in $\mathcal{H}$ that defines the closed string kinetic term, namely $\frac{1}{2}\langle\Psi, Q_c\Psi\rangle_c$. Finally, we let $Q = Q_o + Q_c$. We then have

$$\{Q, f(\mathcal{A})\} = -f(\partial\mathcal{A})\,, \tag{5.9}$$

as expected from the analogous closed string relation [6].

We can now simply set

$$S = Q + f(\mathcal{V})\,, \tag{5.10}$$

and it follows immediately from (5.8) and (5.9) that

$$\tfrac{1}{2}\{S, S\} + \hbar\Delta S = -f\Big(\partial\mathcal{V} + \tfrac{1}{2}\{\mathcal{V}, \mathcal{V}\} + \hbar\Delta\mathcal{V}\Big) \tag{5.11}$$

where use was made of $\Delta Q = 0$ [17]. It is now clear that the master equation (3.2) satisfied by the moduli space $\mathcal{V}$ guarantees that the master action satisfies the BV master equation.

6. Vertices coupling open and closed strings via a disk

In this section we will study in detail the simplest couplings of open strings to closed strings, namely, couplings via a disk. The Riemann surfaces are therefore disks with punctures on the boundary, representing the open strings, and punctures in the interior, representing the closed strings. We will give the complete specification of the minimal area metric for these surfaces. This will give us insight into the way the minimal area problem works, and in addition it will make our discussion very concrete. In all our analysis we will have in mind the classical open string field theory of Witten.

These vertices are of particular interest since they enter in the construction of a global closed string symmetry of open string field theory (sect.7), and in the construction of an open string field theory in a nontrivial closed string background (sect.8).

We will begin by discussing some algebraic aspects of the coupling of a closed string to a boundary, and then some algebraic properties of the open-closed vertex, in particular maps of cohomologies (the string diagrams corresponding to these two vertices were described earlier.). We then turn to the open-open-closed vertex, which is also examined in detail. It is then possible to consider the case of M open strings coupling to a single closed string. We then turn to the case of two closed strings coupling to an open string (this vertex actually vanishes in the light-cone gauge). After studying very explicitly this case, we are able to generalize to the case of M-open strings coupling to two closed strings, and then to the case of M open strings coupling to N closed strings.

6.1. Closed String coupling to a empty boundary

The vertex was described before in section 4.1 (Fig.8). The Fock space state associated to this vertex is called the boundary state. If we denote by $\mathcal{H}_c$ the vector space of the closed string field, the boundary state $\langle B|$ should be thought as an element of $\mathcal{H}_c^*$ the dual vector space to $\mathcal{H}_c$. Here we will only make some remarks on the alternative ways of describing the boundary state.

We may recall that if H_{cft} denotes the state space of the CFT describing the closed string, then the subspace $\mathcal{H}_c$

$$\mathcal{H}_c = \{|\Psi\rangle \in H_{cft}\,\big|b_0^-|\Psi\rangle = 0,\, L_0^-|\Psi\rangle = 0\}$$

is the vector space associated to the closed string field. Its dual space is therefore described as the space of equivalence relations

$$\mathcal{H}_c^* = \{\langle\Psi| \in H^*;\ \langle\Psi| \sim \langle\Psi| + \langle\Lambda|b_0^-,\ \langle\Psi| \sim \langle\Psi| + \langle\Lambda'|L_0^-\ .\} \tag{6.1}$$

Note that for any state in $\mathcal{H}_c^*$ one can always find a representative that is annihilated by c_0^- and L_0^-. Therefore one can choose $\langle B|$ to satisfy $\langle B|c_0^- = 0$ and $\langle B|L_0^- = 0$. On the other hand, strictly speaking, the fact that the boundary state is annihilated by Q_c just means

$$\langle B|Q_c = 0,\ \text{on}\,\mathcal{H}_c^* \Leftrightarrow \langle B|\,Q_c = \langle *|b_0^-\ , \tag{6.2}$$

for some state $\langle *|$. In general the boundary state $\langle B|$ can couple to every physical state

in $\mathcal{H}_c$. This can be seen as follows. We can rewrite

$$\langle B|\Psi\rangle = {}_3\langle B|\langle\omega_{12}^c|S_{23}\rangle|\Psi\rangle_1 = \langle B,\Psi\rangle_c \quad \text{with} \quad |B\rangle_1 = {}_2\langle B|S_{12}\rangle\,, \tag{6.3}$$

with $|S_{12}\rangle = b_0^-|R'_{12}\rangle$ the inverse of the closed string symplectic form. Note that (6.2) implies that $Q_c|B\rangle = 0$. Since the closed string inner product $\langle\cdot,\cdot\rangle_c$ is nondegenerate [17], for every physical $|\Psi\rangle$ one can find some state $|B\rangle$ that gives a nonzero inner product. Typically, the restriction of the bilinear form to physical states is nondegenerate, and in this case the state $|B\rangle$ can be chosen to be physical (see, for example, appendix C of [15]). Since the ghost number of $|\Psi\rangle$ is two, and the bilinear form couples states of ghost numbers adding to five, $|B\rangle$ must be of ghost number three. Therefore, *the physical content of the boundary closed string vertex is defined by a state in the cohomology of Q_c at ghost number three.*

In bosonic string theory the physical cohomology sits at ghost number two, but a complete copy of this cohomology sits at ghost number three. At ghost number two, all physical states with the exception of the ghost dilaton, are of the form $c_1\bar{c}_1|m\rangle$, where $|m\rangle$ denotes state built of matter operators. For these states, their ghost number three physical counterparts are of the form $(c_0+\bar{c}_0)c_1\bar{c}_1|m\rangle$. Even for the ghost dilaton $|D_gh\rangle = (c_1c_{-1} - \bar{c}_1\bar{c}_{-1})|0\rangle$ its ghost number three counterpart is obtained by multiplication with $c_0+\bar{c}_0$. It is known, however, that if one extends suitably the complex $\mathcal{H}_c$ to include the zero modes x of the bosonic coordinates, all cohomology at ghost number three vanishes identically [28,29]. This reflects the fact that one can remove the one point functions of the closed string physical fields by giving space-time dependent expectation values to background fields.

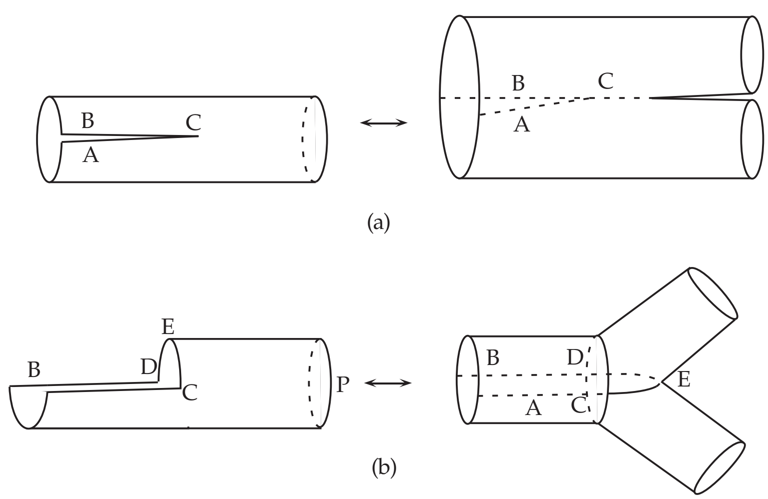


Figure 12. (a) An open-closed light-cone vertex, the open string endpoints lie on A and B. At C the endpoints join to make a closed string. The open-closed light-cone vertex when doubled gives us the three closed string light cone vertex. (b) The double of the covariant open-closed vertex gives the standard symmetric three closed string vertex.

6.2. The Open-Closed String Vertex

Light-cone field theory has an open-closed string vertex. It corresponds to an open string of some fixed length closing up to make a closed string of the same length [4]. Its double corresponds to a three-closed string light-cone vertex (Fig. 12). The covariant open closed vertex was examined earlier and is shown again in Fig. 12(b). Its double, shown to the right, is the three closed string diagram.

This vertex is denoted in the operator formalism as the bra $\langle\mathcal{V}_{oc}| \in \mathcal{H}_o^* \otimes \mathcal{H}_c^*$, and the general arguments of [30] imply that it is annihilated by the sum of BRST operators:

$$\langle\mathcal{V}_{oc}|\,(\,Q^o + Q^c\,) = 0. \tag{6.4}$$

This vertex couples open string states $|\Phi\rangle$, to closed string states $|\Psi\rangle$. Since the surface has no conformal Killing vectors the sum of the respective ghost numbers must be three.⋆ The term in the string action is given by $\mathcal{L}_{oc} \sim \langle\mathcal{V}_{oc}|\Phi\rangle|\Psi\rangle$, and it should be noted that ghost number allows the coupling of classical open strings (ghost number one) to classical closed strings (ghost number two).

The open-closed string vertex naturally defines two maps; one from $\mathcal{H}_o$ to $\mathcal{H}_c^*$ and the other from $\mathcal{H}_c$ to $\mathcal{H}_o^*$. By composing these maps with the maps taking the dual spaces to the spaces we can obtain a map of the open string state space to the closed string state space, and a map from the closed string state space to the open string state space (these maps are not inverses!). Given an open string state $|\Phi\rangle$, we obtain the closed string state

$$|\Psi(\Phi)\rangle = \langle\mathcal{V}_{oc}|\Phi\rangle|S^c\rangle, \tag{6.5}$$

where $|S^c\rangle = b_0^-|R_c\rangle$ is the inverse of the closed string symplectic form. Since $|S^c\rangle$ is of ghost number five, it follows that if the ghost number of $|\Phi\rangle$ is p, that of $|\Psi\rangle$ is $p+2$. Thus we write

$$\mathcal{V}_{0c} : \mathcal{H}_o^{(*)} \to \mathcal{H}_c^{(*+2)}\,. \tag{6.6}$$

In a completely analogous way we also find that

$$\mathcal{V}_{0c} : \mathcal{H}_c^{(*)} \to \mathcal{H}_o^{(*)}\,. \tag{6.7}$$

These maps are dependent on the choices of local coordinates defining the open-closed string vertex. In other words, they take different forms for the light-cone and the covariant vertex. It follows from (6.4) that these maps induce maps in cohomology; indeed, BRST trivial open states are mapped to BRST trivial closes string states (recall that $(Q_c^{(1)} + Q_c^{(2)})|S_{12}^c\rangle = 0$). We thus have

$$\begin{aligned} \mathcal{V}_{0c} :\ H_o^{(*)} &\to H_c^{(*+2)} \\ \mathcal{V}_{0c} :\ H_c^{(*)} &\to H_o^{(*)}\,. \end{aligned} \tag{6.8}$$

The above maps in cohomology are quite interesting and it would be nice to know more about them. For BRST classes that have primary representatives, the above maps are independent of the particular choice of open-closed string vertex. The kernels of the above maps will play some role in sections 8 and 9.

⋆ We are assigning zero ghost number to both the SL(2,R) and SL(2,C) vacua.

6.3. The Open-Open-Closed String Vertex

In order to extract this vertex let us consider the scattering of two open strings and a closed string off a disk. As before, the surface can be thought as a unit disk with the closed string puncture at the origin. The two open string punctures can be put symmetrically with respect to the real axis, as shown in Fig. 13. The angle θ shown in the figure, must vary from zero to π in order to represent the complete moduli space. In addition to an elementary interaction, the only possible Feynman graph for the process in question is one built with a three open string vertex and an open-closed vertex, as shown in Fig. 14.

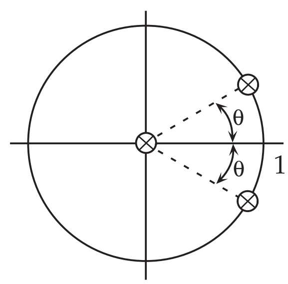


Figure 13. A disk with two punctures on the boundary and one in the interior, representing the coupling of two open strings to a closed string. The open string punctures are placed symmetrically with respect to the real axis, and the closed string puncture is placed at the origin. The full moduli space is described by $0 \le \theta \le \pi$.

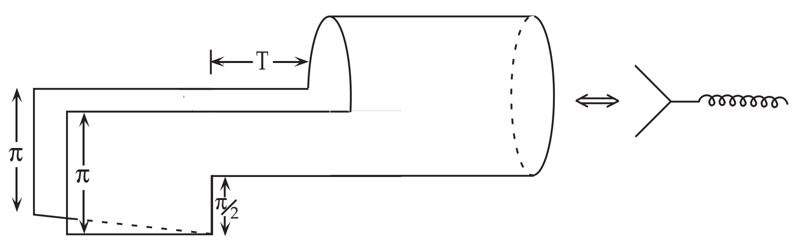


Figure 14. A Feynman graph built with a three open string vertex, an open string propagator of length T, and an open-closed vertex.

As the open string propagator becomes infinitely long, namely, $T \to \infty$, the two open string punctures are getting close to each other, $\theta \to 0$ (Fig. 13). When the propagator collapses, $T = 0$, the angle $\theta = \theta_0$ is less than $\pi/2$. This is so because from the viewpoint of the closed string the open strings are not opposite to each other. The same Feynman graph with the punctures exchanged covers the interval $\pi - \theta_0 \le \theta \le \pi$. The missing region of moduli space is therefore $\theta_0 < \theta < \pi - \theta_0$. This region must be generated by the open-open-closed elementary vertex shown in Fig. 15. Note that the two open strings ovelap with each other along the segment AB, taken to be of length a_{12}. In the endpoint configuration ($T = 0$) for the Feynman graph of Fig. 14, $a_{12} = \pi/2$. In fact, a_{12} cannot exceed $\pi/2$; if it did, the nontrivial open curve CAD shown in Fig. 15(a), would be shorter than π: $l_{CAD} = l_{CA} + l_{AD} \le 2 \cdot \pi/2 = \pi$. The interaction vertex corresponds to the region $a_{12} \in [0, \pi/2]$. The pattern of overlaps is indicated in Fig. 15(b). When $a_{12} = 0$, the

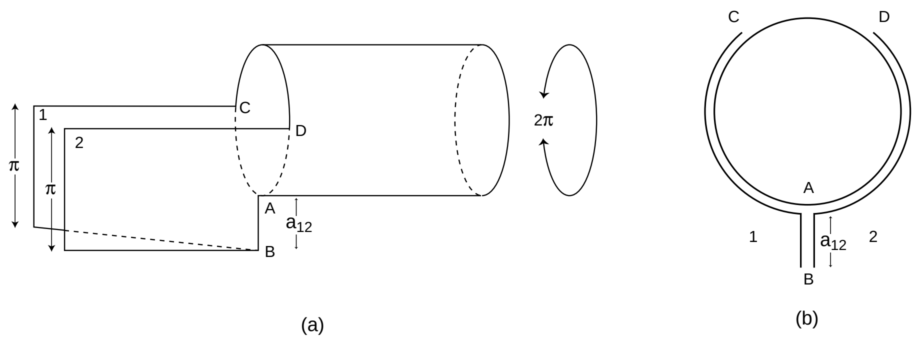


Figure 15.The interaction vertex for two open strings and one closed string. The two open strings have an overlap AB of length a_{12}. The rest of the open strings create part of the emerging closed string. This interaction has one modular parameter a_{12}, which must be smaller than $\pi/2$, since otherwise the nontrivial open curve CAD would be shorter than π.

open strings cover the closed string and are opposite to each other. This configuration corresponds to $\theta = \pi/2$.

6.4. The Vertex Coupling M Open Strings to a Closed String

Here we consider the case of M open strings and a single closed string scattering off a disk. The surfaces building the vertex for this process fall into two different types. In the first type the open strings surround completely the closed string. In the second type the open strings do not surround completely the closed string; there are M such terms, correponding to the M configurations compatible with the cyclic ordering of the open strings.

For both types, the only nontrivial closed curves are those going around the closed string puncture, and all those are manifestly longer than 2π. Thus the only relevant constraints are the ones on open curves. Consider Fig. 16, the condition that the nontrivial open curve $AA'B'C'C$ be longer or equal to π implies that $a_{ij} \leq \pi/2$. In fact, this is the unique constraint, all open string overlap segments must be shorter or equal to $\pi/2$. The reader can verify that all other nontrivial open curves are then automatically longer or equal to π.

How about boundary matching? The only inequalities that restrict the region of integration for the vertex are $0 \leq a_{ij} \leq \pi/2$. Boundaries may only arise when these inequalities are saturated. Whenever $a_{ij} = 0$, there is actually no boundary. Consider the configuration where the open strings surround completely the closed string. Whenever an overlap segment becomes zero, the configuration turns smoothly into one of the configurations that do not surround fully the closed string. Whenever an overlap segment becomes zero in one of the latter configurations one actually loses all modular parameters of the configuration, so this is not a relevant boundary. One can understand this by examining Fig. 17(a). Segments such as AB or $A'B'$ must be at least of length $\pi/2$. The same is true for segments CD and $C'D'$. These four segments therefore cover fully the closed string. The only way this is possible is if all these four segments measure exactly $\pi/2$, the points A and A' coincide, and every other a_{ij} is identically equal to $\pi/2$, as shown to the right

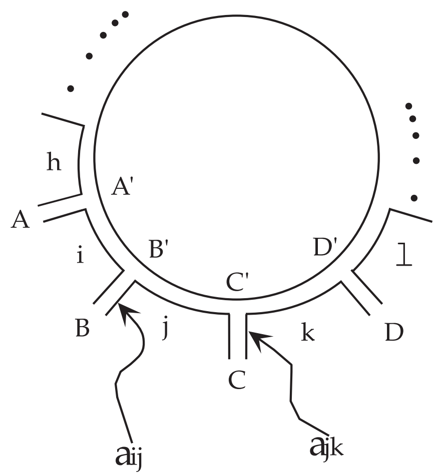


Figure 16. A generic configuration coupling several open strings to a single closed string. The necessary and sufficient condition in order to have all nontrivial open curves longer or equal to π is that all open string overlaps a_{ij} be shorter than or equal to $\pi/2$.

in the figure. Indeed, this configuration has no modular parameters. It also follows from this argument that there is no relevant configuration where the set of open strings breaks into two clusters, since each cluster must at least cover a length π in the closed string.

Whenever $a_{ij} = \pi/2$ one does get a boundary. Such configuration matches smoothly with a Feynman graph having strings i and j replaced by a single string k that is connected via a propagator to a three string vertex having strings i and j as external strings. As illustrated in Fig. 17(b), when the propagator collapses ($T = 0$) the resulting configuration matches with the vertex boundary. It is clear that for any possible collapsed propagator configuration there is a corresponding vertex boundary configuration. Thus the matching of boundaries is complete.

Note that the vertex has $(M-1)$ modular parameters. This is most easily seen from the configuration where the open strings surround fully the closed string. In this case the M parameters a_{ij} must satisfy a single equality: $\pi M - 2\sum a_{ij} = 2\pi$.

6.5. The Open-Closed-Closed Vertex

The vertices coupling two closed strings to open strings via a disk are relevant for the elucidation of the algebra of global symmetries induced by closed strings. We discuss next the simplest such vertex, having two closed strings and a single open string. This vertex fills a region of moduli space of dimension two, and the generic surface is shown in Fig. 18. The two closed strings, labeled 1 and 2, overlap across the arc BD, and create a closed boundary $\mathcal{C}$, part of which is attached to the open string. The corresponding overlap pattern is shown to the right. Note that the arc BD must be shorter than π, otherwise the closed curve $\mathcal{C}$ would be shorter than 2π. The length of BD is thus $(\pi - \alpha)$, with $\alpha \geq 0$. The curve $\mathcal{C}$ has length greater or equal to 2π. The open string must overlap with both closed strings, as shown in the figure, if it did not, the arc BD, which is shorter than π, would become a nontrivial open arc, thus violating a length condition. The length of

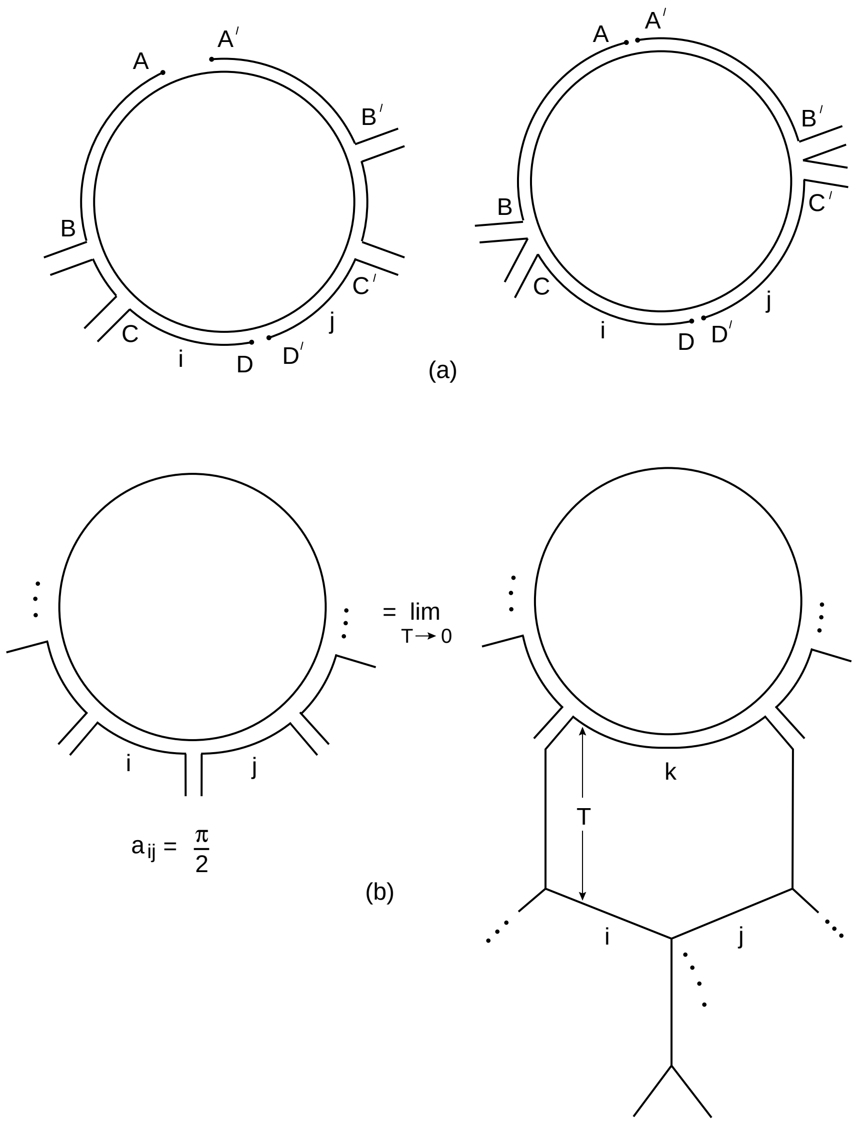


Figure 17. (a) A configuration where an overlap a_{ij} vanishes, is not a codimension one boundary. It can be shown that it has no modular parameters, and must really look like the diagram to the right. (b) When some a_{ij} becomes $\pi/2$, the vertex shown to the left matches with the $T = 0$ limit of the Feynman graph shown to the right. This Feynman graph uses a vertex coupling $(M - 1)$ open strings to a closed string, and a three open string vertex.

the segment AB, where the open string and closed string number one overlap, is denoted by $\beta \geq 0$.

There are two open paths whose length must be checked:

$$l_{ABD} = l_{AB} + l_{BD} = \beta + (\pi - \alpha) \geq \pi \quad \rightarrow \quad \alpha \leq \beta, \tag{6.9}$$

$$l_{CBD} = l_{CB} + l_{BD} = (\pi - \beta) + (\pi - \alpha) \geq \pi \quad \rightarrow \alpha + \beta \leq \pi. \tag{6.10}$$

These two inequalities, together with $\alpha, \beta \geq 0$, make the region indicated in Fig. 19(a). This is actually half the moduli space, the other half corresponds to the open string glued

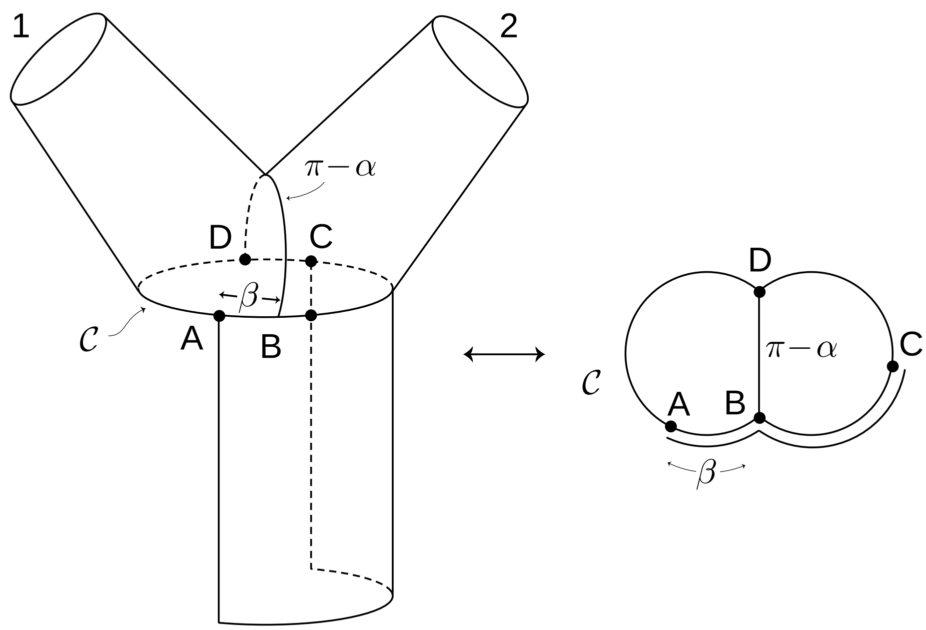


Figure 18. The covariant vertex for closed-closed-open strings. The two closed strings overlap across the arc BD of length $(\pi - \alpha)$, with $\alpha \geq 0$. The closed curve $\mathcal{C}$ going through A, B, C, D, and back to A, is longer than 2π. The open string emerges to the bottom of the diagram. To the right we show the pattern of overlaps in this interaction.

to $\mathcal{C}$ such that point B is on the boundary, and point D is not.

What do the boundaries of the region in Fig. 19(a) match to? Consider first the boundary $\alpha = 0$. In this configuration the two closed strings are overlapping like in the three closed string vertex. Indeed, the length of $\mathcal{C}$ is precisely 2π. This boundary matches with a Feynman graph where we have a three closed string vertex joining to an closed-open vertex via a closed string propagator, in the limit when this closed string propagator collapses and becomes the curve $\mathcal{C}$.

The other two boundaries appear from the same type of process: the appearance of an intermediate open string propagator. They correspond to the nontrivial open curves ABD and CBD considered above becoming of length π. When $l_{ABD} = \pi$ an open strip can grow separating closed string 1 from the rest of the interaction. The Feynman graph it corresponds to is one where a open-closed vertex turning closed string 1 into an open string is joined by an open string propagator to an open-open-closed vertex. As indicated in Fig. 19(b), the open string to the left closes up to make closed string number one. This configuration is recognized to have $\alpha = \beta$. Analogous remarks apply to l_{CBD} becoming of length π (Fig. 19(c)).

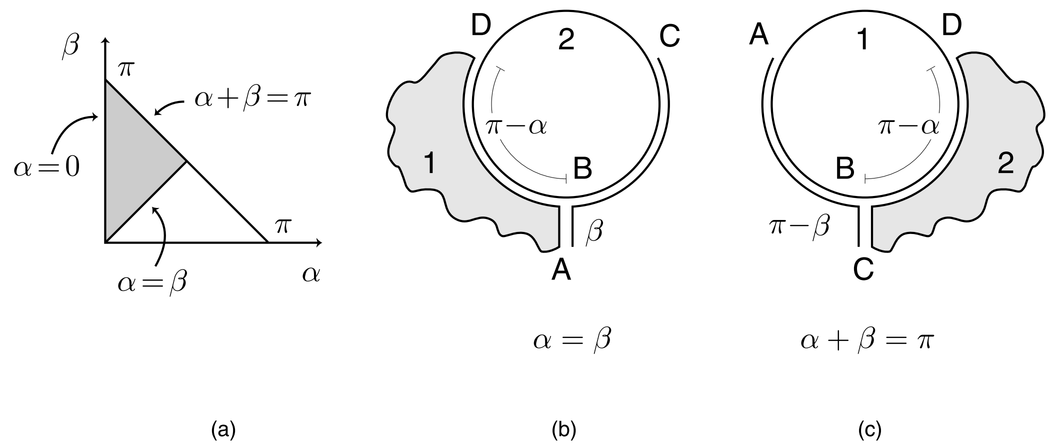


Figure 19. (a) The parameter region for α and β in the closed-closed-open vertex shown in Fig. 12. There are three boundaries. When $\alpha = 0$ the nontrivial closed curve $\mathcal{C}$ becomes of length 2π. (b) This configuration matches the $\alpha = \beta$ boundary of the parameter region. It corresponds to the nontrivial open curve ABD becoming of length π. (c) This configuration matches the $\alpha + \beta = \pi$ boundary of the parameter region. It corresponds to the nontrivial open curve CBD becoming of length π.

6.6. Vertex Coupling M Open Strings to two Closed Strings

The geometrical arrangement for this coupling follows from the previous case together with that of the coupling of many open strings to a single closed string. As in the previous case, the two closed strings join via an arc of length less than π, and form a closed curve $\mathcal{C}$, as was the case in Fig. 18. The M open strings then couple as if $\mathcal{C}$ was a closed string. There is just one qualitative difference; since the 'closed string' $\mathcal{C}$ is longer than 2π the cluster of M open strings can break into subclusters. In the present case, since the maximum possible length of $\mathcal{C}$ is 4π, the open strings can break nontrivially into two or three separate clusters, each of which must cover at least a length π on $\mathcal{C}$. One must still require that all overlaps a_{ij} between open strings be less than or equal to $\pi/2$. There are additional constraints, however, corresponding to nontrivial open curves that use the arc joining the two closed strings

The counting of modular parameters is simple if one refers to the case when the open strings surround $\mathcal{C}$ completely. There are M parameters for the open string overlaps, one for the length of $\mathcal{C}$, and one constraint relating them, giving M parameters. There is one extra parameter coming from a global rotation of the cluster of open strings along $\mathcal{C}$. This gives a total of $M + 1$ modular parameters.

The boundaries are of two types, boundaries matching to Feynman graphs with a closed string propagator and boundaries matching to Feynman graphs with an open string propagator. The first type arises when the closed curve $\mathcal{C}$ becomes of length 2π; the Feynman graph is built from a three closed string vertex and a vertex coupling one closed string to M open strings. The second type falls into two cases. In the first one an overlap a_{ij} of open strings becomes of length $\pi/2$, and we get an open string propagator separating away open strings i and j (see Fig. 17(b)). In the second one, a nontrivial open curve that includes the arc separating the two closed strings, becomes of length π. Here an open

string propagator can grow separating out one of the closed strings and some number of open strings.

6.7. Vertices Coupling N Closed Strings to M Open Strings

This is the most general coupling via a disk. Thanks to the practice gained in the previous examples it can be dealt with easily. The N closed strings make up a polyhedron with $(N+1)$ faces. Each of the N faces corresponding to the closed strings is of perimeter 2π. The remaining face is left open, we call its boundary $\mathcal{C}$, and it is a closed curve of length greater or equal to 2π. All closed paths in this polyhedron must be longer or equal to 2π. The number of modular parameters of this polyhedron is one more than that of a standard $(N+1)$-faced polyhedron, because the open face has no perimeter constraint. Since the latter has $2(N+1)-6$ parameters, this gives $2N-3$ modular parameters for the open-faced polyhedron.

The cluster of M open strings is attached to the closed curve $\mathcal{C}$, just as in the previous case. This cluster can again break into several clusters, depending on the length of $\mathcal{C}$. For fixed length of C, the open strings add $(M-1)+1$ parameters, where the last one comes from twist. The total number of modular parameters is therefore $2N+M-3 = 2(N-1)+(M-1)$.

There are three types of nontrivial closed curves on the polyhedron: (a) closed curves that do not use any arc of $\mathcal{C}$, (b) closed curves that use pieces of $\mathcal{C}$, and (c) the closed curve $\mathcal{C}$. When any of these curves becomes of length 2π we match to a Feynman graph with an intermediate closed string propagator. Both for cases (a) and (b) the diagram breaks into a pure closed string vertex coupling a subset of the N external closed strings, and a vertex coupling the remaining closed strings to all of the M open strings. For case (c) the diagram breaks into a pure $N+1$ closed string vertex and a vertex coupling one closed string to all of the open strings. Note that for all closed string intermediate channels the open strings must remain together since otherwise one would get two boundary components.

There are two types of nontrivial open curves: (i) curves that do not use arcs joining closed strings, and (ii) curves that do. The first type of curves saturate when an overlap a_{ij} becomes of length $\pi/2$, and we get an open string propagator separating away open strings i and j (see Fig. 17(b)). When type (ii) curves saturate, an open string intermediate strip separates the vertex into two vertices each carrying a nonzero number of closed strings and some number of external open strings.

7. Open string symmetries of closed string origin

In this section we discuss symmetries of classical open string theory that arise from the closed string sector. Such symmetries were first discussed by Hata and Nojiri [16] in the context of covariantized light-cone string field theories. Due to the much improved understanding of the recursion relations of string vertices and of BV quantization of string theory, our discussion will be quite transparent. We will also be able to discuss the relation between these trasformations and the usual open string gauge transformations, as well as elucidating the algebra of such symmetries.

7.1. Symmetry transformations

We are dealing with symmetries of the classical open string action. We recall that this action can be written as

$$S_o = Q_o + f(\mathcal{V}_0)\,, \tag{7.1}$$

where $\mathcal{V}_0$ is the formal sum of all the string vertices coupling open strings only through a disk. The consistency of this action follows from the recursion relation (3.17) which reads

$$\partial\mathcal{V}_0 + \tfrac{1}{2}\{\mathcal{V}_0\,,\,\mathcal{V}_0\} = 0\,. \tag{7.2}$$

We now consider the symmetry transformation generated by the hamiltonian

$$U = f_{\mathcal{O}}(\mathcal{V}_1)\,, \tag{7.3}$$

where $\mathcal{O}$ is an arbitrary closed string state *annihilated* by Q_c, and $\mathcal{V}_1$ is the sum of string vertices coupling all numbers of open strings to a single closed string through a disk. We recall that these vertices satisfy the recursion relation (3.19) which reads

$$\partial\mathcal{V}_1 + \{\mathcal{V}_0\,,\,\mathcal{V}_1\} = 0 \tag{7.4}$$

The variation of the action under an infinitesimal transformation generated by U is given by

$$\begin{aligned}\delta S_o = \{S_o, U\} &= \Big\{Q_o + f(\mathcal{V}_0)\,, f_{\mathcal{O}}(\mathcal{V}_1)\Big\}\\ &= -f_{\mathcal{O}}\Big(\partial\mathcal{V}_1 + \{\mathcal{V}_0\,,\,\mathcal{V}_1\}\Big) - f_{Q_c\mathcal{O}}(\mathcal{V}_1)\end{aligned} \tag{7.5}$$

where we added and subtracted a term involving the closed string BRST operator in order to produce the boundary action on $\mathcal{V}_1$. It is now evident that δS_o vanishes by virtue of the recursion relation (7.4) and the condition $Q_c|\mathcal{O}\rangle = 0$. This establishes that (7.3) generates a symmetry. The open string field variation induced by this hamiltonian is simply

$$\delta|\Phi\rangle = \{|\Phi\rangle, U\} \sim \frac{\partial\, U}{\partial\,|\Phi\rangle}\,|S^o\rangle\,, \tag{7.6}$$

and given that $U \sim \langle\mathcal{V}_{oc}|\mathcal{O}\rangle|\Phi\rangle + \cdots$, we have that

$$\delta|\Phi\rangle \sim \langle\mathcal{V}_{oc}|\mathcal{O}\rangle|S^o\rangle + \cdots\,. \tag{7.7}$$

The symmetry transformation, to first approximation, shifts the open string field along the image of the closed string state $|\mathcal{O}\rangle$ under the map (6.7) defined by $\mathcal{V}_{oc}$. The nonlinear terms of the transformation involve one or more open string fields and use the vertices coupling several open strings to a single closed string.

Since the open closed vertex maps closed string states of a given ghost number to open string states of the same ghost number (see (6.7)) BRST closed states of ghost number one in the closed string sector are of particular interest. Such states would induce nontrivial transformations in the physical ghost number one sector of the open string field. Indeed we know that closed string BRST classes at ghost number one correspond to global symmetries of the closed string sector. This point has been made very precise by defining the proper BRST cohomology taking into account zero modes of the bosonic coordinates [28,29]. In the bosonic context Poincare transformations arise from the closed string BRST cohomology at ghost number one. The corresponding open string transformation defines the induced Poincare transformation of the open string field.

We wish to examine what happens when the closed string state $|\mathcal{O}\rangle$ used to define the open string transformation is Q_c-trivial. We will show that in this case we simply get a gauge transformation of the open string sector. This confirms that only nontrivial Q_c classes generate new transformations. Recall that open string gauge transformations are generated by Hamiltonians U of the form

$$U = \{S_o, \Lambda\}\,. \tag{7.8}$$

Choose now $\Lambda = f_\eta(\mathcal{V}_1)$ where η is a closed string state. We then find

$$\begin{aligned} U &= \{S_o, f_\eta(\mathcal{V}_1)\} \\ &= \{Q_o + f(\mathcal{V}_0)\,, f_\eta(\mathcal{V}_1)\} \\ &= -f_{Q_c\eta}(\mathcal{V}_1) - f_\eta\Big(\partial\mathcal{V}_1 + \{\mathcal{V}_0\,,\,\mathcal{V}_1\}\Big) \\ &= -f_{Q_c\eta}(\mathcal{V}_1)\,. \end{aligned} \tag{7.9}$$

Comparing the last and first lines of the above equation we see that indeed, an open string transformation induced by $Q_c\eta$ can be written as an open string gauge transformation induced by the gauge parameter $f_\eta(\mathcal{V}_1)$.

7.2. The Algebra of Symmetry Transformations

Let us now examine the algebra of closed string induced open-string transformations. The result is simple: the commutator of two such transformations gives another closed string induced transformation, and an open string gauge transformation. To show this consider two Q_c-closed string states $|\mathcal{O}_1\rangle$ and $|\mathcal{O}_2\rangle$ and the associated hamiltonians $U_1 = f_{\mathcal{O}_1}(\mathcal{V}_1)$ and $U_2 = f_{\mathcal{O}_2}(\mathcal{V}_1)$. The hamiltonian defining the commutator of two such transformations is simply the antibracket of the two hamiltonians. We find

$$\{U_1\,,\,U_2\} = \{f_{\mathcal{O}_1}(\mathcal{V}_1)\,,\,f_{\mathcal{O}_2}(\mathcal{V}_1)\} = -f_{\mathcal{O}_1\mathcal{O}_2}\Big(\{\mathcal{V}_1\,,\,\mathcal{V}_1\}_o\Big)\,, \tag{7.10}$$

where the subscript in the antibracket reminds us that sewing is only performed in the open string sector (this must be the case since we cannot lose the two closed string punctures). We now recall the recursion relation (3.21) involving the open-closed vertices $\mathcal{V}_2$ having

two closed string insertions

$$\partial\mathcal{V}_2 + \{\mathcal{V}_0, \mathcal{V}_2\} + \tfrac{1}{2}\{\mathcal{V}_1, \mathcal{V}_1\}_o + \{\mathcal{V}_1, \mathcal{V}_3^c\} = 0\,. \tag{7.11}$$

Back in (7.10) we get

$$\begin{aligned}\{U_1\,,\,U_2\} &= 2f_{\mathcal{O}_1\mathcal{O}_2}\Big(\partial\mathcal{V}_2 + \{\mathcal{V}_0, \mathcal{V}_2\}\Big) + 2f_{\mathcal{O}_1\mathcal{O}_2}\Big(\{\mathcal{V}_1, \mathcal{V}_3^c\}\Big) \\ &= -2\Big\{S_o\,,\, f_{\mathcal{O}_1\mathcal{O}_2}(\mathcal{V}_2)\Big\} - 2\Big\{f(\mathcal{V}_1)\,,\, f_{\mathcal{O}_1\mathcal{O}_2}(\mathcal{V}_3^c)\Big\}\,,\end{aligned} \tag{7.12}$$

where we made use of the fact that the states $|\mathcal{O}_1\rangle$ and $|\mathcal{O}_2\rangle$ are Q_c closed. The first term in the last right hand side is recognized as a hamiltonian inducing an open string gauge transformation. The second term is recognized as being of the form $f_\Omega(\mathcal{V}_1)$ where the closed string state Ω is the state induced by inserting $|\mathcal{O}_1\rangle$ and $|\mathcal{O}_2\rangle$ on two of the punctures of the three closed string vertex $\mathcal{V}_3^c$. More precisely $|\Omega\rangle = \langle\mathcal{V}_3^c|\mathcal{O}_1\rangle|\mathcal{O}_2\rangle|S\rangle$. This confirms the claim for the commutator of two closed string-induced open string symmetries.

The commutator of a closed string-induced symmetry transformation and a open string gauge transformation is an open string gauge transformation. This is easily verified,

$$\Big\{f_{\mathcal{O}}(\mathcal{V}_1)\,, \{S_o\,, \lambda\}\Big\} \sim \Big\{S_o\,, \{\lambda\,, f_{\mathcal{O}}(\mathcal{V}_1)\}\Big\} + \Big\{\lambda\,, \{f_{\mathcal{O}}(\mathcal{V}_1)\,, S_o\}\Big\}\,, \tag{7.13}$$

by use of the Jacobi identity of the antibracket. The last term in the right hand side vanishes ((7.5)) and the first term indeed represents the hamiltonian for an open string gauge transformation.

8. Open string theory on closed string backgrounds

This time we are interested in a truncation of the full open-closed theory to a classical open string sector in interaction with an explicit closed string background. The classical open string field theory of (7.1) is one such system, and indeed it is classically gauge invariant. This action is obtained by setting to zero the fluctuating closed string field in the open-closed string theory action. It thus represents open string theory on the closed string background described by the conformal theory having Q_c as the BRST operator. We now imagine giving the closed string field an expectation value $|\Psi_0\rangle$ thus changing the closed string background in a controlled way. What we will show is that we can obtain a consistent classical open string theory whenever $|\Psi_0\rangle$ satisfies the closed string field equation of motion. This action represents open string theory on the closed string background described by $|\Psi_0\rangle$.

We claim that the open-string action is given as

$$S_o(\Phi, \Psi) = Q_o + \sum_{k=0}^{\infty} \mathcal{V}_k \equiv Q_o + f(\overline{\mathcal{V}})\,, \tag{8.1}$$

where we have included now all couplings of open and closed strings through disks. This action depends both on the open string field and the closed string field, but we will think

of the closed string field as non-dynamical. For consistency this action should satisfy the classical master equation. We compute

$$\{S_o\,,\,S_o\}_o = 2\{Q_o\,,f(\overline{\mathcal{V}})\} + \{f(\overline{\mathcal{V}})\,,\,f(\overline{\mathcal{V}})\}_o\,, \tag{8.2}$$

where the antibracket has been restricted to the open string sector, given that the closed string field is not dynamical. Now add and subtract a closed string BRST operator to find

$$\begin{aligned}\{S_o\,,\,S_o\} &= 2\{Q_o + Q_c\,,f(\overline{\mathcal{V}})\} + \{f(\overline{\mathcal{V}})\,,\,f(\overline{\mathcal{V}})\}_o - 2\{Q_c\,,f(\overline{\mathcal{V}})\}\\ &= -2f\Big(\partial\overline{\mathcal{V}} + \tfrac{1}{2}\{\overline{\mathcal{V}}\,,\,\overline{\mathcal{V}}\}_o\Big) - 2\{Q_c\,,f(\overline{\mathcal{V}})\}\,,\end{aligned} \tag{8.3}$$

In order to proceed we must recall the recursion relation (3.23) satisfied by the $\overline{\mathcal{V}}$ spaces

$$\partial\overline{\mathcal{V}} + \tfrac{1}{2}\{\overline{\mathcal{V}}\,,\,\overline{\mathcal{V}}\}_o + \{\overline{\mathcal{V}}\,,\,\mathcal{V}^c\} = 0\,, \tag{8.4}$$

where $\mathcal{V}^c$ is the sum of pure closed string vertices on the sphere. Back in (8.3) we obtain

$$\begin{aligned}\{S_o\,,\,S_o\} &= 2\,f\,(\{\mathcal{V}^c\,,\,\overline{\mathcal{V}}\}) - 2\{Q_c\,,f(\overline{\mathcal{V}})\}\,,\\ &= -2f\Big(\{Q_c + f(\mathcal{V}^c)\,,\,f(\overline{\mathcal{V}})\,\}\Big)\,,\\ &= -2f\,(\{S_c\,,\,f(\overline{\mathcal{V}})\,\}\,)\,,\end{aligned} \tag{8.5}$$

where in the last step we have recognized the classical closed string action. Since

$$\{S_c\,,\,f(\overline{\mathcal{V}})\,\} \sim \frac{\partial\,S_c}{\partial|\Psi\rangle}\,\frac{\partial\,f(\overline{\mathcal{V}})}{\partial\,|\Psi\rangle}\,|S\rangle\,, \tag{8.6}$$

this term vanishes for closed string fields satisfying the closed string field equation of motion. In this case we find $\{S_o, S_o\}_o = 0$ as we wanted to show. This verifies the consistency of this generalized classical open string field theory. As usual, the master equation guarantees the gauge invariance of the theory, and we have that hamiltonians of the type $U = \{S_o, \Lambda\}_o$ generate gauge transformations. Such gauge transformations depend on the closed string background, since S_o depends on the closed string background.

The reverse construction, trying to describe a closed string sector moving on an open string background, does not seem possible. The obvious ansatz $S_c = Q_c + f(\mathcal{V}_c) + f(\overline{\mathcal{V}})$ fails. Attempts made using other subalgebras of surfaces, such as that of all genus zero surfaces (see (3.24)) failed as well.

9. On the ghost-dilaton theorem and background independence of open-closed theory

In this section we sketch briefly the main issues that arise when one attempts to establish a ghost-dilaton theorem in open-closed string theory. We also comment on the similar issues that arise in trying to prove background independence.

It is useful, for clarity, to write out explicitly the first few terms of the open-closed string action. To this end we use the list of vertices apppearing at the first few orders of $\hbar$ as given in section 3.2. We have, without attention to dimensionless coefficients,

$$\begin{aligned} S \sim\, & \big(\langle\Phi, Q_o\Phi\rangle_o + \kappa\Phi^3 + \kappa^2\Phi^4 + \cdots\big) + \langle\Psi, Q_c\Psi\rangle_c \\ & + \hbar^{\frac{1}{2}}\big(\kappa^2\Psi^3_{sph} + \Psi_{disk} + \kappa(\Psi\,\Phi)_{disk} + \kappa^2(\Psi\,\Phi^2)_{disk} + \kappa^3(\Psi\,\Phi^3)_{disk}\cdots\big) \\ & + \hbar\big(\kappa^4\Psi^4_{sph} + Z_{torus} + \kappa^2\Psi^2_{disk} + \kappa^3(\Psi^2\Phi)_{disk} + \cdots + Z_{ann} + \kappa\Phi_{ann} + \cdots\big) \\ & + \hbar^{\frac{3}{2}}\big(\kappa^6\Psi^5_{sph} + \kappa^2\Psi_{torus} + \kappa^4\Psi^3_{disk} + \kappa^5(\Psi^3\Phi)_{disk} + \cdots\big) \\ & + \cdots \end{aligned} \tag{9.1}$$

Here the subscripts indicate the type of surface involved in the vertex, with *sph* standing for sphere and *ann* standing for annulus.

In order to establish a ghost dilaton theorem, as that proven for closed strings in Refs.[31,32], we have to show that there is a dilaton hamiltonian U_D that generates a string field redefinition that changes the string coupling in the action. In particular, at the full quantum level, the dilaton hamiltonian must satisfy

$$\kappa\frac{dS}{d\kappa} = \{S, U_D\} + \hbar\Delta U_D\,, \tag{9.2}$$

as discussed in section 4 of Ref.[31]. The dilaton hamiltonian U_D must begin with a term of the type $\langle D, \Psi\rangle_c$ that gives a shift in the closed string field along the dilaton direction $|D\rangle$. The shift must be of the form

$$|\Psi\rangle \to \frac{1}{\hbar^{1/2}\kappa^2}|D\rangle + |\Psi\rangle\,, \tag{9.3}$$

where we have inserted the proper factors of $\hbar$ and string coupling κ necessary for the shift to have the correct effect. Indeed, we see in (9.1) that each closed string insertion contributes a prefactor of $\hbar^{1/2}\kappa^2$ to the vertices. This shift will not only change the string coupling as it appears in purely closed string terms, but also ought to change the string coupling in terms involving open strings. It certainly has the potential to do so; for example, the term $\kappa(\Phi^3)_{disk}$ can have its coupling changed due to the term $\hbar^{1/2}\kappa^3(\Phi^3\Psi)_{disk}$. There are a few terms, however, that introduce interesting complications.

The coupling $\langle\Psi, B\rangle_c$ of the closed string to the boundary state (section 6.1) indicated above as Ψ_{disk}, and appearing at order $\hbar^{1/2}$ with no coupling constant dependence is one of those terms. When we shift by the ghost dilaton, we get a constant term of the form

$$\frac{1}{\kappa^2}\hbar^0\langle D, B\rangle_c\,, \tag{9.4}$$

This constant has the dependence to be interpreted as arising from a disk with no punctures (which is precisely what we get when the closed string puncture is erased). This

implies that we must include in the above action a *tree level cosmological term* arising as an open string partition function on the disk

$$S = \frac{1}{\kappa^2}\, Z_{disk} + \cdots \ , \tag{9.5}$$

since otherwise (9.3) could not be satisfied.[⋆] Note that the effect of shifting along the dilaton producing (9.4) is due to the first term in the right hand side of (9.3). The second term in (9.3) can also produce a constant, but its geometrical interpretation indicates that it refers to surfaces of higher genus or surfaces with more than one boundary component. The overlap $\langle D, B\rangle_c$ of the ghost-dilaton with the boundary state does not vanish in general. In fact, it does not vanish for any oriented open string theory. It can vanish in non-oriented open string theories where there is a similar contribution arising from a closed string ending on a crosscap. This cancellation only happens when the gauge group of the open string is $SO(8192)$ [34]. As we see here, only for this gauge group we expect the tree level cosmological constant to vanish.

A second term that raises some puzzles is the term $\hbar^{1/2}\kappa(\Psi\Phi)_{disk}$, involving the open-closed vertex $\langle \mathcal{V}_{oc}|$. When the closed string field is shifted by the dilaton we now get a term linear in the open string field

$$\frac{1}{\kappa}\langle \mathcal{V}_{oc}|D\rangle|\Phi\rangle\,, \tag{9.6}$$

Since such a term cannot exist in the resulting open-closed string action, it must be cancelled by giving the open string field a vacuum expectation value

$$|\Phi\rangle \to |\Phi\rangle + \frac{1}{\kappa}|d\rangle\,. \tag{9.7}$$

This shift of the kinetic term produces a linear term that cancels (9.6) provided that

$$Q_o|d\rangle \sim \langle \mathcal{V}_{oc}|D\rangle|S^o\rangle\,. \tag{9.8}$$

The right hand side of this equation is simply the image of the ghost dilaton under the open-closed map (section 6.2). This equation can be solved for $|d\rangle$ only if the open-closed map takes the cohomology class of $|D\rangle$ to zero, or in other words, to a trivial open string state. This is the case, as can be verified by using $|D\rangle = Q_c|\chi\rangle$, where $|\chi\rangle = c_0^-|0\rangle$.[†] One readily finds that $|d\rangle \sim \langle \mathcal{V}_{oc}|\chi\rangle|S^o\rangle$ and therefore the open string shift is of the form

$$|\Phi\rangle \to |\Phi\rangle + \frac{1}{\kappa}\,\langle \mathcal{V}_{oc}|\chi\rangle|S^o\rangle\,, \tag{9.9}$$

a shift along an unphysical direction.

⋆ In early discussions of these issues in the framework of effective actions, terms that could yield cosmological constants were attributed to open string loop corrections to closed string beta functions [33].

† Note that the ghost-dilaton $|D\rangle$ is not trivial because $|\chi\rangle = c_0^-|0\rangle$ is not a state in the closed string complex. We can nevertheless use this equation here because $\mathcal{V}_{oc}$ is well defined in the complete state space $\mathcal{H}_{cft}$.

Rather similar issues arise in a background independence analysis. If we shift along a closed string direction specified by a dimension zero primary state V of ghost number two, we may also generate a constant term from the closed-boundary interaction. This will have the interpretation of a change in the partition function of the disk. Moreover, a linear term in the open string field can arise. There should be no difficulty in removing it by an unphysical shift in the open string. This follows because the open closed map would map closed string cohomology at ghost number two to open string cohomology at ghost number two, and such cohomology is known to vanish in the appropriate complex where we include the zero modes of the non-compact bosonic coordinates [28].

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REFERENCES

1. J. Polchinski, *Tasi Lectures on D- branes*, hep-th/9611050.
2. E. Witten, *Noncommutative geometry and string field theory*, Nucl. Phys. **B268** (1986) 253.
3. D. Z. Freedman, S. B. Giddings, J. A. Shapiro, and C. B. Thorn,*The nonplanar one-loop amplitude in Witten's string field theory*, Nucl. Phys. **B287** (1987) 61
4. J. A. Shapiro and C. B. Thorn, *Closed string-open string transitions and Witten's string field theory*, Phys. Lett **B194** (1987) 43
5. B. Zwiebach, *Quantum open string theory with manifest closed string factorization*, Phys. Lett. **B256** (1991) 22
6. A. Sen and B. Zwiebach, *Quantum background independence of closed string field theory* Nucl. Phys.**B423** (1994) 580, hep-th/9311009.
7. A. Sen and B. Zwiebach, *Background independent algebraic structures in closed string field theory*, Comm. Math. Phys. **177** (1996) 305, hep-th/9408053.
8. B. Zwiebach, *New moduli spaces from background independence consistency conditions*, Nucl. Phys. **B480** (1996) 507, hep-th/9605075
9. B. Zwiebach, *String field theory around non-conformal backgrounds* , Nucl. Phys. **B480** (1996) 541, hep-th/9606153
10. A. Schwarz, *Grassmannians and string theory*, hep-th/9610122.
11. T. R. Morris, Nucl. Phys. **B297** (1988) 141;
J. L. Mañes, Nucl. Phys. **B303** (1988) 305;
M. Maeno, Phys. Lett. **B216** (1989) 81;
R. Potting and C. Taylor, Nucl. Phys. **B316** (1989) 59;
T. Kugo, Prog. Theor. Phys. **78** (1987) 690;
S. J. Sin, Nucl. Phys. **B306** (1988) 282; B313 (1989) 165;
W. Siegel and B. Zwiebach, Nucl. Phys. **B282** (1987) 125;
P. Goddard, J. Goldstone, C. Rebbi and C. B. Thorn, Nucl. Phys. **B56** (1973) 109

12. Y. Saitoh and Y. Tanii, Nucl. Phys. **B325** (1989) 161; Nucl. Phys. **B331** (1990) 744.
13. K. Kikkawa and S. Sawada, *Cancellation of the Lorentz anomaly of string field theory in light-cone gauge*, Nucl. Phys. **B335** (1990) 677.
14. J.D. Stasheff, *On the homotopy associativity of H-spaces, I.*, Trans. Amer. Math. Soc. **108**, 275 (1963); *On the homotopy associativity of H-spaces, II.*, Trans. Amer. Math. Soc. **108**, 293 (1963).
15. M. Gaberdiel and B. Zwiebach, *Tensor constructions of Open String Theories I: Foundations*, hep-th/9705038.
16. H. Hata and M. M. Nojiri, *New symmetry in covariant open-string field theory*, Phys. Rev **D36** (1987) 1193
17. B. Zwiebach, 'Closed string field theory: Quantum action and the Batalin-Vilkovisky master equation', Nucl. Phys **B390** (1993) 33, hep-th/9205075.
18. H. Sonoda and B. Zwiebach, *Closed string field theory loops with symmetric factorizable quadratic differentials*, Nucl. Phys. B331 (1990) 592.
19. B. Zwiebach, *Quantum closed strings from minimal area*, Mod. Phys. Lett. A5 (1990) 2753.
20. B. Zwiebach, *How covariant closed string theory solves a minimal area problem*, Comm. Math. Phys. **136** (1991) 83; *Consistency of closed string polyhedra from minimal area*, Phys. Lett. **B241** (1990) 343
21. B. Zwiebach, *A proof that Witten's open string theory gives a single cover of moduli space*, Comm. in Math. Phys. **142** (1991) 193.
22. B. Zwiebach, *Minimal area problems and quantum open strings* , B. Zwiebach, Comm. in Math. Phys.**141** (1991) 557.
23. B. Zwiebach, *Interpolating string field theories*, Mod. Phys. Lett. **A7** (1992) 1079, hep-th/9202015.
24. H. Hata and B. Zwiebach, *Developing the covariant Batalin-Vilkovisky approach to string theory* , Ann. Phys. **229** (1994) 177, hep-th/9301097.
25. C. B. Thorn, *String field theory*, Phys. Rep. **174** (1989) 1.
26. A. R. Bogojevic, *BRST Invariance of the measure in string field theory*, Phys. Lett. **B198** (1987) 479
27. M. Maeno, *Canonical quantization of Witten's string field theory using midpoint light cone time* Phys. Rev. **D43** (1991) 4006
28. A. Astashkevich and A. Belopolsky, *String center of mass operator and its effect on BRST cohomology*, to appear in Comm. Math. Phys. hep-th/9511111.
29. A. Belopolsky and B. Zwiebach, *Who changes the string coupling ?* Nucl. Phys. **B472** (1996) 109, hep-th/9511077.
30. L. Alvarez-Gaume, C. Gomez, G. Moore and C. Vafa, Nucl. Phys. **B303** (1988) 455

31. O. Bergman and B. Zwiebach, *The dilaton theorem and closed string backgrounds*, Nucl. Phys. **B441** (1995) 76, hep-th/9411047

32. S. Rahman and B. Zwiebach, *Vacuum vertices and the ghost-dilaton* Nucl. Phys. **B471** (1996) 233, hep-th/

33. C. Callan, C. Lovelace, C. Nappi and S. Yost, *Adding holes and crosscaps to the superstring*, Nucl. Phys. **293** (1987) 83.

34. M. Douglas and B. Grinstein, *Dilaton tadpole for the open bosonic string* Phys. Lett. $\mathcal{B}$183 (1987) 52;
N. Marcus and A. Sagnotti, *Group theory from quarks at the ends of strings*, Phys. Lett. **B188** (1987) 58.